Journal of Number Theory 132 (2012) 1519–1553

Contents lists available at SciVerse ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Modular equations for cubes of the Rogers–Ramanujan and Ramanujan–Göllnitz–Gordon functions and their associated continued fractions

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ARTICLE INFO

Article history:
Received 8 August 2011
Accepted 4 January 2012
Available online 5 March 2012
Communicated by David Goss

MSC:
primary 11A55, 33D90
secondary 11P83

Keywords:
Rogers–Ramanujan functions
Rogers–Ramanujan continued fraction
Ramanujan–Göllnitz–Gordon functions
Ramanujan–Göllnitz–Gordon continued fraction
Ramanujan's notebooks
Modular identities
Theta functions
Partitions
Colored partitions

ABSTRACT

In this paper, we prove modular identities involving cubes of the Rogers–Ramanujan functions. Applications are given to proving relations for the Rogers–Ramanujan continued fraction. Some of our identities are new. We establish analogous results for the Ramanujan–Göllnitz–Gordon functions and the Ramanujan–Göllnitz–Gordon continued fraction. Finally, we offer applications to the theory of partitions.

1. Introduction

Here and in the sequel, we assume that $|q| < 1$, and employ the standard product notation

$$(a;q)_0 := 1, \qquad (a;q)_n := \prod_{i=0}^{n-1}\left(1 - aq^i\right) \quad (n \geqslant 1),$$

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0022-314X/$ – see front matter
doi:10.1016/j.jnt.2012.01.005

and

$$(a;q)_\infty := \prod_{i=0}^{\infty}\left(1-aq^i\right).$$

The famous Rogers–Ramanujan functions are defined by

$$G(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} \quad \text{and} \quad H(q) := \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n}. \tag{1.1}$$

These functions satisfy the well-known Rogers–Ramanujan identities [32,26], [28, pp. 214–215]

$$G(q) = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty} \quad \text{and} \quad H(q) = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}. \tag{1.2}$$

At the end of his brief communication [27], [28, p. 231] announcing his proofs of the Rogers–Ramanujan identities (1.2), Ramanujan remarks, "I have now found an algebraic relation between $G(q)$ and $H(q)$, viz:

$$H(q)\left\{G(q)\right\}^{11} - q^2 G(q)\left\{H(q)\right\}^{11} = 1 + 11q\left\{G(q)H(q)\right\}^6. \tag{1.3}$$

Another noteworthy formula is

$$H(q)G\left(q^{11}\right) - q^2 G(q)H\left(q^{11}\right) = 1.$$

Each of these formulae is the simplest of a large class." Ramanujan did not indicate how he had proved these identities. They are two identities from a list of 40 identities for $G(q)$ and $H(q)$ that he never published.

In the years since, the 40 identities have been established in a series of papers by L.J. Rogers [33] in 1919, G.N. Watson [38] in 1933, D. Bressoud [12,13] in 1977, A.F.J. Biagioli [9] in 1989, and most recently H. Yesilyurt [39,40] in 2009 and 2010. It should be remarked that the complete list was not brought before the mathematical public until 1975 when B.J. Birch [10] found them in the Oxford University library. Although all the identities can be proved using the theory of modular forms, the method employed by Biagioli, it is more instructive to find proofs that Ramanujan might have found. Outside the theory of modular forms, Roger's method, which was generalized by Bressoud and also extended by Yesilyurt, is the only general method that has been devised for proving identities from Ramanujan's list. For details, proofs, and references, see the excellent monograph [8] of B.C. Berndt, G. Choi, Y.-S. Choi, H. Hahn, B.P. Yeap, A.J. Yee, H. Yesilyurt, and J. Yi [8].

Most of the identities in Ramanujan's list involve the Rogers–Ramanujan functions in combinations involving terms of the form

$$G\left(q^\alpha\right)H\left(q^\beta\right) - q^{\frac{\alpha-\beta}{5}} G\left(q^\beta\right)H\left(q^\alpha\right)$$

or

$$G\left(q^\alpha\right)G\left(q^\beta\right) + q^{\frac{\alpha+\beta}{5}} H\left(q^\alpha\right)H\left(q^\beta\right).$$

Curiously, only one of Ramanujan's 40 identities, namely (1.3), involves powers of the Rogers–Ramanujan functions. Nevertheless, Ramanujan's remark that "Each of these formulae is the simplest of a large class" strongly suggests that there are further identities of interest involving powers of the

Rogers–Ramanujan functions. In this paper, we prove identities involving third powers of the Rogers–Ramanujan functions. This work continues the work of the author in [23] and [24], in which identities for squares, fourth, and fifth powers of the Rogers–Ramanujan functions are studied. Moreover, we apply our results to prove identities for the famous Rogers–Ramanujan continued fraction, defined by

$$R(q) := \frac{q^{1/5}}{1} \, + \, \frac{q}{1} \, + \, \frac{q^2}{1} \, + \, \frac{q^3}{1} \, + \cdots.$$

The advantage of our method is that we are able to offer new identities for the Rogers–Ramanujan continued fraction; some of our identities yield for the first time closed-form product representations for factors appearing in certain known modular equations for the Rogers–Ramanujan continued fraction.

Two important analogues of the Rogers–Ramanujan functions are the Ramanujan–Göllnitz–Gordon functions, defined by

$$S(q) := \sum_{n=0}^{\infty} \frac{(-q;q^2)_n}{(q^2;q^2)_n} q^{n^2} \quad \text{and} \quad T(q) := \sum_{n=0}^{\infty} \frac{(-q;q^2)_n}{(q^2;q^2)_n} q^{n^2+2n}.$$

The functions $S(q)$ and $T(q)$ satisfy relations analogous to the Rogers–Ramanujan identities, namely,

$$S(q) = \frac{1}{(q;q^8)_\infty (q^4;q^8)_\infty (q^7;q^8)_\infty} \quad \text{and} \quad T(q) = \frac{1}{(q^3;q^8)_\infty (q^4;q^8)_\infty (q^5;q^8)_\infty}. \tag{1.4}$$

Identities (1.4) were first recorded by Ramanujan on p. 41 of his Lost Notebook. They can also be found in L.J. Slater's list [34, Eqs. (36), (34)], but with q replaced by $-q$. Identities (1.4) are the analytic versions of the combinatorial Göllnitz–Gordon identities [21,22].

In addition to the product-series identities (1.2), and (1.4), the Rogers–Ramanujan and Ramanujan–Göllnitz–Gordon functions share further remarkable properties. For instance, S.-S. Huang [25] has derived an extensive list of elegant modular relations for $S(q)$ and $T(q)$ analogous to Ramanujan's list of 40 identities for the Rogers–Ramanujan functions. S.-L. Chen and Huang [17] expanded on this list. Subsequently, N.D. Baruah, J. Bora, and N. Saikia [4] offered new proofs of many of the identities of Chen and Huang; their methods yielded further new relations as well.

Rogers [32] proved that the Rogers–Ramanujan continued fraction and the Rogers–Ramanujan functions are intimately connected by the relation

$$R(q) = q^{1/5} \frac{(q;q^5)_\infty (q^4;q^5)_\infty}{(q^2;q^5)_\infty (q^3;q^5)_\infty} = q^{1/5} \frac{H(q)}{G(q)}. \tag{1.5}$$

Likewise, the Ramanujan–Göllnitz–Gordon continued fraction, $K(q)$, defined by

$$K(q) := \frac{q^{1/2}}{1+q} \, + \, \frac{q^2}{1+q^3} \, + \, \frac{q^4}{1+q^5} \, + \, \frac{q^6}{1+q^7} \, + \cdots,$$

has an analogous product representation in terms of the Ramanujan–Göllnitz–Gordon functions. Indeed, on p. 229 of his second notebook [29], Ramanujan recorded that

$$K(q) = q^{1/2} \frac{(q;q^8)_\infty (q^7;q^8)_\infty}{(q^3;q^8)_\infty (q^5;q^8)_\infty} = q^{1/2} \frac{T(q)}{S(q)}. \tag{1.6}$$

Independently, Göllnitz [21] and Gordon [22] rediscovered and proved (1.6). Shortly thereafter, Andrews [1] proved (1.6) as a corollary of a more general result. The theory of the Ramanujan–Göllnitz–Gordon continued fraction has been further developed in recent years by various authors, including H.H. Chan and Huang [16]; K.R. Vasuki and B.R. Srivatsa Kumar [37]; and B. Cho, J.K. Koo, and Y.K. Park [18].

In this paper, after establishing results for cubes of the Rogers–Ramanujan functions and consequent identities for the Rogers–Ramanujan continued fraction, we establish an analogous theory for $S(q)$, $T(q)$, and $K(q)$. Many of our identities are new; in particular, to the author's knowledge, there are no identities for cubes of the Ramanujan–Göllnitz–Gordon functions previously stated in the literature.

In the final section of our paper, we apply certain of our identities to derive theorems in the theory of partitions.

2. Preliminaries

Ramanujan's general theta function is defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \qquad |ab| < 1. \tag{2.1}$$

Basic properties satisfied by $f(a,b)$ include [5, p. 34, Entry 18]

$$f(a,b) = f(b,a), \tag{2.2}$$

$$f(1,a) = 2f\left(a,a^3\right), \tag{2.3}$$

$$f(-1,a) = 0, \tag{2.4}$$

and, if n is an integer,

$$f(a,b) = a^{n(n+1)/2} b^{n(n-1)/2} f\left(a(ab)^n, b(ab)^{-n}\right). \tag{2.5}$$

Property (2.2) is used frequently and without mention in the sequel. The function $f(a,b)$ satisfies the well-known Jacobi Triple Product Identity [5, Entry 19, p. 35],

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{2.6}$$

The three most important special cases of (2.6) are [5, p. 36, Entry 22]

$$\phi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \left(-q;q^2\right)_\infty^2 \left(q^2;q^2\right)_\infty, \tag{2.7}$$

$$\psi(q) := f\left(q,q^3\right) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{2.8}$$

and

$$f(-q) := f\left(-q,-q^2\right) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty. \tag{2.9}$$

The latter equality in (2.9) is Euler's pentagonal number theorem. Note that $q^{1/24} f(-q) = \eta(\tau)$, where $q = e^{2\pi i\tau}$, $\operatorname{Im}\tau > 0$, and $\eta(\tau)$ is the Dedekind-eta function. Also, after Ramanujan, define

$$\chi(q) := \left(-q; q^2\right)_\infty. \tag{2.10}$$

We record here a useful lemma, which follows from the product representations (2.7)–(2.10).

Lemma 2.1.

$$\phi(-q) = \frac{f^2(-q)}{f(-q^2)}, \qquad \psi(q) = \frac{f^2(-q^2)}{f(-q)}, \qquad \chi(-q) = \frac{f(-q)}{f(-q^2)},$$
$$\phi(q) = \frac{f^5(-q^2)}{f^2(-q)f^2(-q^4)}, \qquad \psi(-q) = \frac{f(-q)f(-q^4)}{f(-q^2)},$$
$$\chi(q) = \frac{f^2(-q^2)}{f(-q)f(-q^4)}, \qquad f(q) = \frac{f^3(-q^2)}{f(-q)f(-q^4)}.$$

The following is a general theta function identity due to Ramanujan.

Lemma 2.2. *(See [5, p. 48, Entry 31].) For each nonnegative integer n, let $U_n = a^{n(n+1)/2} b^{n(n-1)/2}$ and $V_n = a^{n(n-1)/2} b^{n(n+1)/2}$. Then*

$$f(U_1, V_1) = \sum_{r=0}^{n-1} U_r f\left(\frac{U_{n+r}}{U_r}, \frac{V_{n-r}}{U_r}\right). \tag{2.11}$$

Lemma 2.3 follows from the addition of the two identities in [5, p. 45, Entry 29].

Lemma 2.3. *If $ab = cd$, then*

$$f(a,b)f(c,d) = f(ac,bd)f(ad,bc) + af\left(\frac{b}{c}, \frac{c}{b}abcd\right)f\left(\frac{b}{d}, \frac{d}{b}abcd\right). \tag{2.12}$$

Next, we state the important Quintuple Product Identity, a version of which can be found on p. 207 of Ramanujan's Lost Notebook.

Lemma 2.4. *(See [5, p. 80, Eq. (38.2)], [2, p. 14, Entry 1.2.1].)*

$$f\left(B^3 q, q^5/B^3\right) - B^2 f\left(q/B^3, B^3 q^5\right) = f\left(-q^2\right)\frac{f(-B^2, -q^2/B^2)}{f(Bq, q/B)}. \tag{2.13}$$

We require the following elementary results for the Rogers–Ramanujan and Ramanujan–Göllnitz–Gordon functions. The identities are consequences of (2.6), (1.2), (2.1), (2.9), (1.4), and (2.8).

Lemma 2.5. *We have*

$$G(q) = \frac{f(-q^2, -q^3)}{f(-q)}, \qquad H(q) = \frac{f(-q, -q^4)}{f(-q)}, \tag{2.14}$$

and

$$G(q)H(q) = \frac{f(-q^5)}{f(-q)}. \tag{2.15}$$

Lemma 2.6. *We have*

$$S(q)=\frac{f(-q^3,-q^5)}{\psi(-q)},\qquad T(q)=\frac{f(-q,-q^7)}{\psi(-q)}, \tag{2.16}$$

and

$$S(q)T(q)=\frac{f(-q^2)f^2(-q^8)}{f(-q)f^2(-q^4)}. \tag{2.17}$$

H. Schröter [5, pp. 65–72] developed useful representations for a product of two theta functions as a sum of m products of pairs of theta functions. An elegant generalization of Schröter's work has been discovered by R. Blecksmith, J. Brillhart, and I. Gerst [11, Theorem 2]. Recently, Z. Cao [14,15] has proved a generalization of Schröter's work that includes as a special case the theorem of Blecksmith, Brillhart, and Gerst. Below, we translate [11, Theorem 2] into Ramanujan's notation. Define, for $\epsilon\in\{0,1\}$ and $|ab|<1$,

$$f_\epsilon(a,b)=\sum_{n=-\infty}^{\infty}(-1)^{\epsilon n}(ab)^{n^2/2}(a/b)^{n/2}.$$

Theorem 2.7. *(See [11, Theorem 2].) Let a, b, c, and d denote positive numbers with $|ab|$, $|cd|<1$. Suppose that there exist positive integers α, β, and m such that*

$$(ab)^\beta=(cd)^{\alpha(m-\alpha\beta)}.$$

Let $\epsilon_1,\epsilon_2\in\{0,1\}$, and define $\delta_1,\delta_2\in\{0,1\}$ by

$$\delta_1\equiv\epsilon_1-\alpha\epsilon_2\pmod 2\quad\textit{and}\quad\delta_2\equiv\beta\epsilon_1+p\epsilon_2\pmod 2,$$

respectively, where $p=m-\alpha\beta$. Then, if R denotes any complete residue system modulo m,

$$\begin{aligned}&f_{\epsilon_1}(a,b)f_{\epsilon_2}(c,d)\\&\quad=\sum_{r\in R}(-1)^{\epsilon_2 r}c^{r(r+1)/2}d^{r(r-1)/2}f_{\delta_1}\left(\frac{a(cd)^{\alpha(\alpha+1-2r)/2}}{c^\alpha},\frac{b(cd)^{\alpha(\alpha+1+2r)/2}}{d^\alpha}\right)\\&\qquad\times f_{\delta_2}\left(\frac{(b/a)^{\beta/2}(cd)^{p(m+1-2r)/2}}{c^p},\frac{(a/b)^{\beta/2}(cd)^{p(m+1+2r)/2}}{d^p}\right).\end{aligned}$$

3. Auxiliary theta function identities

The following theta function identity was stated by Ramanujan and first proved by Berndt [5, p. 349, Entry 2(ii)]. Subsequently, Baruah and Bora [3, Theorem 3.6], working on modular equations of degree 9, provided a different proof. Below, we offer a new proof that shows that (3.1) may be viewed as a consequence of the Quintuple Product Identity (2.13).

Lemma 3.1. *We have*

$$\psi(q)-3q\psi\left(q^9\right)=\frac{\phi(-q)}{\chi(-q^3)}. \tag{3.1}$$

Proof. By [5, p. 49, Corollary (ii)], we know that

$$\psi(q) = f\left(q^3, q^6\right) + q\psi\left(q^9\right).$$

Hence, applying (2.8) and (2.3), we deduce that

$$\begin{aligned}\psi(q) - 3q\psi\left(q^9\right) &= f\left(q^3, q^6\right) - 2q\psi\left(q^9\right)\\ &= f\left(q^3, q^6\right) - qf\left(1, q^9\right). \end{aligned}\tag{3.2}$$

In the Quintuple Product Identity (2.13), replace q by $q^{3/2}$ and then set $B = q^{1/2}$ to find that

$$f\left(q^3, q^6\right) - qf\left(1, q^9\right) = f\left(-q^3\right)\frac{f(-q, -q^2)}{f(q, q^2)}.\tag{3.3}$$

Applying (2.9) six times, (2.6), and Lemma 2.1, we deduce that

$$\begin{aligned} f\left(-q^3\right)\cdot\frac{f(-q, -q^2)}{f(q, q^2)} &= \left(q^3; q^3\right)_\infty \cdot \frac{f(-q)}{(-q; q^3)_\infty(-q^2; q^3)_\infty(q^3; q^3)_\infty}\\ &= f(-q)\cdot\frac{(-q^3; q^3)_\infty}{(-q; q)_\infty}\\ &= f(-q)\cdot\frac{(q; q)_\infty}{(q^2; q^2)_\infty}\cdot\frac{(q^6; q^6)_\infty}{(q^3; q^3)_\infty}\\ &= \frac{f^2(-q)}{f(-q^2)}\cdot\frac{f(-q^6)}{f(-q^3)}\\ &= \frac{\phi(-q)}{\chi(-q^3)}. \end{aligned}\tag{3.4}$$

Employing (3.4) in (3.3) and combining the resulting equation with (3.2), we complete the proof. □

The first identity in the next lemma is [5, p. 379, Entry 10(ii)], and the second is [6, p. 188, Entry 36(ii)].

Lemma 3.2. *Let $A = f(-q^4, -q^{11})$ and $B = qf(-q, -q^{14})$. Then,*

(i) $$A - B = \frac{f(-q, -q^4)}{f(-q^2, -q^3)} f\left(-q^5\right),$$

(ii) $$A^3 - B^3 = \frac{f(-q^3, -q^{12})}{f(-q^6, -q^9)} f^3\left(-q^5\right).$$

In the following two theorems, we record new theta function identities that exhibit a high degree of symmetry. The theorems are applied in Sections 4 and 6, respectively. We believe that the inherent symmetry makes these identities of interest in their own right.

Theorem 3.3. *Define*

$$A := q^6 f\left(q^{12}, q^{78}\right), \qquad B := q^9 f\left(q^3, q^{87}\right), \qquad C := q^4 f\left(q^{18}, q^{72}\right),$$
$$D := f\left(q^{33}, q^{57}\right), \qquad E := f\left(q^{42}, q^{48}\right), \qquad F := qf\left(q^{27}, q^{63}\right).$$

Furthermore, define

$$\bar{A} := q^7 f\left(q^9, q^{81}\right), \qquad \bar{B} := q^8 f\left(q^6, q^{84}\right), \qquad \bar{C} := q^3 f\left(q^{21}, q^{69}\right),$$
$$\bar{D} := f\left(q^{36}, q^{54}\right), \qquad \bar{E} := f\left(q^{39}, q^{51}\right), \qquad \bar{F} := q^2 f\left(q^{24}, q^{66}\right).$$

Then

$$\text{(i)} \qquad AB - BC + CD - DE + EF - FA = -f\left(-q, -q^4\right) f\left(-q^{15}, -q^{30}\right),$$

$$\text{(ii)} \qquad \bar{A}\bar{B} - \bar{B}\bar{C} + \bar{C}\bar{D} - \bar{D}\bar{E} + \bar{E}\bar{F} - \bar{F}\bar{A} = -f\left(-q^2, -q^3\right) f\left(-q^{15}, -q^{30}\right).$$

Proof. Applying (2.11) with $n = 3$, $a = -q$, and $b = -q^4$, we deduce that

$$\begin{aligned} f\left(-q, -q^4\right) &= f\left(-q^{18}, -q^{27}\right) - qf\left(-q^{33}, -q^{12}\right) + q^7 f\left(-q^{48}, -q^{-3}\right) \\ &= f\left(-q^{18}, -q^{27}\right) - qf\left(-q^{33}, -q^{12}\right) - q^4 f\left(-q^3, -q^{42}\right), \end{aligned} \tag{3.5}$$

where in the last line we applied (2.5). Next, we employ (2.12) successively with $a = -q^{18}$, $b = -q^{27}$, $c = -q^{15}$, $d = -q^{30}$; $a = -q^{12}$, $b = -q^{33}$, $c = -q^{15}$, $d = -q^{30}$; and $a = -q^3$, $b = -q^{42}$, $c = -q^{15}$, and $d = -q^{30}$, in order to deduce, respectively, that

$$f\left(-q^{18}, -q^{27}\right) f\left(-q^{15}, -q^{30}\right) = f\left(q^{33}, q^{57}\right) f\left(q^{42}, q^{48}\right) - q^{15} f\left(q^{12}, q^{78}\right) f\left(q^3, q^{87}\right), \tag{3.6}$$

$$f\left(-q^{12}, -q^{33}\right) f\left(-q^{15}, -q^{30}\right) = f\left(q^{27}, q^{63}\right) f\left(q^{42}, q^{48}\right) - q^{12} f\left(q^{18}, q^{72}\right) f\left(q^3, q^{87}\right), \tag{3.7}$$

and

$$f\left(-q^3, -q^{42}\right) f\left(-q^{15}, -q^{30}\right) = f\left(q^{18}, q^{72}\right) f\left(q^{33}, q^{57}\right) - q^3 f\left(q^{27}, q^{63}\right) f\left(q^{12}, q^{68}\right), \tag{3.8}$$

where we made one application of (2.5) in deducing the first line. Combining (3.6)–(3.8) with (3.5), we conclude that

$$\begin{aligned} & f\left(-q, -q^4\right) f\left(-q^{15}, -q^{30}\right) \\ &\quad = f\left(-q^{18}, -q^{27}\right) f\left(-q^{15}, -q^{30}\right) - qf\left(-q^{33}, -q^{12}\right) f\left(-q^{15}, -q^{30}\right) \\ &\qquad - q^4 f\left(-q^3, -q^{42}\right) f\left(-q^{15}, -q^{30}\right) \\ &\quad = \left(f\left(q^{33}, q^{57}\right) f\left(q^{42}, q^{48}\right) - q^{15} f\left(q^{12}, q^{78}\right) f\left(q^3, q^{87}\right)\right) \\ &\qquad - q\left(f\left(q^{27}, q^{63}\right) f\left(q^{42}, q^{48}\right) - q^{12} f\left(q^{18}, q^{72}\right) f\left(q^3, q^{87}\right)\right) \\ &\qquad - q^4\left(f\left(q^{18}, q^{72}\right) f\left(q^{33}, q^{57}\right) - q^3 f\left(q^{27}, q^{63}\right) f\left(q^{12}, q^{78}\right)\right) \\ &\quad = DE - AB - FE + CB - CD + FA. \end{aligned} \tag{3.9}$$

Rearranging (3.9), we deduce (i). The proof of (ii) is analogous; we omit the details. □

Remark. Z. Cao has kindly informed the author that the results in Theorem 3.3 may also be deduced as a corollary of a very general theta function identity that he has established [14,15].

Theorem 3.4. *Set*

$$A := q^6 f\left(q^{30}, q^{114}\right), \qquad B := q^{14} f\left(q^6, q^{138}\right), \qquad C := q^{10} f\left(q^{18}, q^{126}\right),$$
$$D := q^2 f\left(q^{42}, q^{102}\right), \qquad E := f\left(q^{66}, q^{78}\right), \qquad F := f\left(q^{54}, q^{90}\right).$$

Furthermore, set

$$\bar{A} := q^{12} f\left(q^{12}, q^{132}\right), \qquad \bar{B} := q^4 f\left(q^{36}, q^{108}\right), \qquad \bar{C} = f\left(q^{60}, q^{84}\right).$$

Then

(i) $$AB - BC + CD - DE + EF - FA = f\left(-q^{24}\right)\psi\left(-q^2\right),$$

(ii) $$\bar{C} - \bar{A} = f\left(-q^{12}\right),$$

(iii) $$\bar{C} + \bar{A} - 2\bar{B} = \frac{\phi(-q^4)}{\chi(-q^{12})},$$

(iv) $$\bar{C}\bar{C} - \bar{A}\bar{A} + 2\bar{B}\bar{A} - 2\bar{B}\bar{C} = f\left(-q^{24}\right)\phi\left(-q^4\right).$$

Proof of (i). By (2.9) and (2.8), the right-hand side of (i) is equal to

$$f\left(-q^{24}, -q^{48}\right) f\left(-q^2, -q^6\right). \tag{3.10}$$

Now apply (2.11) with $n = 3$, $a = -q^2$, and $b = -q^6$, and proceed as in the proof of Theorem 3.3 to deduce (i). □

Proof of (ii). Apply (2.9) and (2.11) with $n = 2$, $a = -q^{12}$, and $b = -q^{24}$. □

Proof of (iii). By (2.8) and (2.11) with $n = 3$, $a = q$, and $b = q^3$, we find that

$$\psi(q) = f\left(q, q^3\right) = f\left(q^{15}, q^{21}\right) + q\psi\left(q^9\right) + q^3 f\left(q^3, q^{33}\right). \tag{3.11}$$

We remark that Ramanujan explicitly recorded (3.11) in his notebooks [5, Corollary (ii), p. 49]. Replacing q by q^4 in (3.11), subtracting $3q^4\psi(q^{36})$ from both sides of the resulting equation, and then applying (2.8), we arrive at

$$\psi\left(q^4\right) - 3q^4\psi\left(q^{36}\right) = f\left(q^{60}, q^{84}\right) + q^{12} f\left(q^{12}, q^{132}\right) - 2q^4\psi\left(q^{36}\right) = \bar{C} + \bar{A} - 2\bar{B}. \tag{3.12}$$

Replacing q by q^4 in Lemma 3.1 and substituting the result into (3.12), we complete the proof. □

Proof of (iv). The product of the left-hand sides of (ii) and (iii) yields the left-hand side of (iv). By Lemma 2.1, the product of the right-hand sides of (ii) and (iii) yields

$$f\left(-q^{12}\right) \cdot \frac{\phi(-q^4)}{\chi(-q^{12})} = f\left(-q^{12}\right) \cdot \frac{f(-q^{24})}{f(-q^{12})} \cdot \phi\left(-q^4\right) = f\left(-q^{24}\right)\phi\left(-q^4\right),$$

which is the right-hand side of (iv). This completes the proof. □

We require one further theorem, namely,

Theorem 3.5. *Define*

$$A := f\left(-q^7, -q^{17}\right), \qquad B := q^2 f\left(-q, -q^{23}\right),$$
$$C := f\left(-q^{11}, -q^{13}\right), \qquad D := qf\left(-q^5, -q^{19}\right).$$

Then,

(i) $$A - B = \frac{f(-q^2, -q^6)}{f(-q^3, -q^5)} f\left(-q^8\right),$$

(ii) $$C + D = \frac{f(-q^2, -q^6)}{f(-q, -q^7)} f\left(-q^8\right),$$

(iii) $$A^3 - B^3 = \frac{f(-q^6, -q^{18})}{f(-q^9, -q^{15})} f^3\left(-q^8\right),$$

(iv) $$C^3 + D^3 = \frac{f(-q^6, -q^{18})}{f(-q^3, -q^{21})} f^3\left(-q^8\right),$$

(v) $$AD + BC = q\frac{f^2(-q^{24})}{f(-q^{12})} f(q),$$

(vi) $$AC + BD = \psi\left(q^3\right) f\left(-q^4\right).$$

Proof of (i), (iii). Let ω denote an arbitrary third root of unity. In the Quintuple Product Identity (2.13), replace q by q^4 and set $B = -\omega q$. We consequently find that

$$\frac{f(-\omega^2 q^2, -\omega q^6)}{f(-\omega q^5, -\omega^2 q^3)} f\left(-q^8\right) = f\left(-q^7, -q^{17}\right) - \omega^2 q^2 f\left(-q, -q^{23}\right) = A - \omega^2 B. \tag{3.13}$$

Setting $\omega = 1$, we immediately deduce (i). Next, note that we obtain one identity from (3.13) for each distinct third root of unity ω. Multiplying together the three identities thus obtained, we find that

$$f^3\left(-q^8\right) \prod_{\omega} \frac{f(-\omega^2 q^2, -\omega q^6)}{f(-\omega q^5, -\omega^2 q^3)} = A^3 - B^3. \tag{3.14}$$

By applying the Jacobi Triple Product Identity (2.6) four times, we deduce that

$$\begin{aligned}\prod_{\omega} f\left(\omega a, \omega^2 b\right) &= f(a, b) f\left(\omega a, \omega^2 b\right) f\left(\omega^2 a, \omega b\right)\\
&= (-a; ab)_\infty (-b; ab)_\infty (-\omega a; ab)_\infty \left(-\omega^2 b; ab\right)_\infty \left(-\omega^2 a; ab\right)_\infty\\
&\quad \times (-\omega b; ab)_\infty (ab; ab)_\infty^3\\
&= \left(-a^3; a^3 b^3\right)_\infty \left(-b^3; a^3 b^3\right)_\infty (ab; ab)_\infty^3\\
&= \frac{f(a^3, b^3) f^3(-ab)}{f(-a^3 b^3)}.\end{aligned} \tag{3.15}$$

We note that (3.15) is also a special case of a general product formula due to S.H. Son [36, Theorem 3.1], [2, p. 14, Lemma 1.2.4]. Employing (3.15) with $a=-q^2$, $b=-q^6$, and with $a=-q^3$, $b=-q^5$, we readily deduce from (3.14) that

$$A^3-B^3=\frac{f(-q^6,-q^{18})}{f(-q^9,-q^{15})}f^3\big(-q^8\big). \tag{3.16}$$

This completes the proof of (iii). □

Proof of (ii), (iv). In (2.13), replace q by q^4 and set $B=-\omega q^3$. Proceeding as in the proofs of (i) and (iii), we easily deduce (ii) and (iv). □

Proof of (v). By (2.6) and (2.9), it is not hard to show that

$$f\big(-q^5,-q^{19}\big)f\big(-q^7,-q^{17}\big)=f\big(-q^5,-q^7\big)\frac{f^2(-q^{24})}{f(-q^{12})} \tag{3.17}$$

and

$$f\big(-q,-q^{23}\big)f\big(-q^{11},-q^{13}\big)=f\big(-q,-q^{11}\big)\frac{f^2(-q^{24})}{f(-q^{12})}. \tag{3.18}$$

Apply (3.17) and (3.18) to discover that

$$\begin{aligned} AD+BC&=qf\big(-q^7,-q^{17}\big)f\big(-q^5,-q^{19}\big)+q^2f\big(-q,-q^{23}\big)f\big(-q^{11},-q^{13}\big)\\ &=q\frac{f^2(-q^{24})}{f(-q^{12})}\big[f\big(-q^5,-q^7\big)+qf\big(-q,-q^{11}\big)\big]. \end{aligned} \tag{3.19}$$

In (2.11), set $a=q$, $b=-q^2$, and $n=2$. Accordingly, we find that

$$f(q)=f\big(q,-q^2\big)=f\big(-q^5,-q^7\big)+qf\big(-q,-q^{11}\big). \tag{3.20}$$

Combining (3.20) with (3.19), we complete the proof. □

Proof of (vi). Set $a=q^3$, $b=q^9$, $c=-q^4$, and $d=-q^8$ in (2.12). Applying also (2.8) and (2.9), we consequently find that

$$\begin{aligned} \psi\big(q^3\big)f\big(-q^4\big)&=f\big(q^3,q^9\big)f\big(-q^4,-q^8\big)\\ &=f\big(-q^7,-q^{17}\big)f\big(-q^{11},-q^{13}\big)+q^3f\big(-q^5,-q^{19}\big)f\big(-q,-q^{23}\big)\\ &=AC+BD. \end{aligned} \tag{3.21}$$

This completes the proof. □

4. Identities with cubes of the Rogers–Ramanujan functions

In this section, we prove identities involving cubes of the Rogers–Ramanujan functions. The identity in Theorem 4.1(i) is originally due to S. Robins [31], who employed the theory of modular forms. A second proof, utilizing elementary methods, has been found by W. Chu [19]. The remaining identities presented in this section are new.

Theorem 4.1. *The following identities hold*:

(i) $$G^3(q)H\big(q^3\big) - G\big(q^3\big)H^3(q) = 3q\frac{f^3(-q^{15})}{f(-q)f(-q^3)f(-q^5)},$$

(ii) $$G^3\big(q^3\big)G(q) + q^2H^3\big(q^3\big)H(q) = \frac{f^3(-q^5)}{f(-q)f(-q^3)f(-q^{15})}.$$

In Ramanujan's list of 40 identities for the Rogers–Ramanujan functions, he recorded several elegant relations for quotients of such identities. For example, he states that [8, p. 10, Eq. (3.35)]

$$\frac{G(q^{17})H(q^2) - q^3H(q^{17})G(q^2)}{G(q^{34})G(q) + q^7H(q^{34})H(q)} = \frac{\chi(-q)}{\chi(-q^{17})}.$$

As an immediate corollary of Theorem 4.1, we have the following new identity involving a quotient of cubic identities for the Rogers–Ramanujan functions.

Corollary 4.2. *The following identity holds*:

$$\frac{G^3(q)H(q^3) - G(q^3)H^3(q)}{G^3(q^3)G(q) + q^2H^3(q^3)H(q)} = 3q\frac{f^4(-q^{15})}{f^4(-q^5)}.$$

We now prove Theorem 4.1. We offer two proofs of (i), and one proof of (ii).

First proof of (i). Applying (2.14), we rewrite (i) in the equivalent form

$$f^3\big(-q^2, -q^3\big)f\big(-q^3, -q^{12}\big) - f^3\big(-q, -q^4\big)f\big(-q^6, -q^9\big) = 3q\frac{f^3(-q^{15})f^2(-q)}{f(-q^5)}. \tag{4.1}$$

We prove (4.1). Let A and B be as in Lemma 3.2. Observe that, by Lemma 3.2,

$$\begin{aligned}
&\frac{1}{f^3(-q^5)}\left(\frac{f(-q^3, -q^{12})}{f(-q^6, -q^9)}f^3\big(-q^5\big) - \frac{f^3(-q, -q^4)}{f^3(-q^2, -q^3)}f^3\big(-q^5\big)\right)\\
&= \frac{1}{f^3(-q^5)}\big(A^3 - B^3 - (A-B)^3\big)\\
&= \frac{1}{f^3(-q^5)}\big(3A^2B - 3AB^2\big)\\
&= \frac{1}{f^3(-q^5)}3AB(A-B)\\
&= 3q\frac{1}{f^3(-q^5)}f\big(-q^4, -q^{11}\big)f\big(-q, -q^{14}\big)\left(\frac{f(-q, -q^4)}{f(-q^2, -q^3)}f\big(-q^5\big)\right).
\end{aligned} \tag{4.2}$$

Adding the fractions on the left-hand side of (4.2), we deduce that

$$\begin{aligned}
&\frac{f^3(-q^2,-q^3)f(-q^3,-q^{12})-f^3(-q,-q^4)f(-q^6,-q^9)}{f(-q^6,-q^9)f^3(-q^2,-q^3)}\\
&\quad=3q\frac{1}{f^2(-q^5)}f\left(-q^4,-q^{11}\right)f\left(-q,-q^{14}\right)\frac{f(-q,-q^4)}{f(-q^2,-q^3)}.
\end{aligned}\tag{4.3}$$

Multiplying (4.3) by $f(-q^6,-q^9)f^3(-q^2,-q^3)$, applying the Jacobi Triple Product Identity (2.6) six times, and then simplifying with the help of (2.9), we conclude from (4.3) that

$$\begin{aligned}
&f^3\left(-q^2,-q^3\right)f\left(-q^3,-q^{12}\right)-f^3\left(-q,-q^4\right)f\left(-q^6,-q^9\right)\\
&\quad=3q\frac{1}{f^2(-q^5)}f\left(-q^4,-q^{11}\right)f\left(-q,-q^{14}\right)f\left(-q,-q^4\right)f\left(-q^6,-q^9\right)f^2\left(-q^2,-q^3\right)\\
&\quad=3q\frac{1}{f^2(-q^5)}\left(q^4;q^{15}\right)_\infty\left(q^{11};q^{15}\right)_\infty\left(q^{15};q^{15}\right)_\infty\left(q;q^{15}\right)_\infty\left(q^{14};q^{15}\right)_\infty\left(q^{15};q^{15}\right)_\infty\\
&\qquad\times\left(q;q^5\right)_\infty\left(q^4;q^5\right)_\infty\left(q^5;q^5\right)_\infty\left(q^6;q^{15}\right)_\infty\left(q^9;q^{15}\right)_\infty\left(q^{15};q^{15}\right)_\infty\\
&\qquad\times\left(q^2;q^5\right)_\infty^2\left(q^3;q^5\right)_\infty^2\left(q^5;q^5\right)_\infty^2\\
&\quad=3q\frac{1}{(q^5;q^5)_\infty^2}\left(q^{15};q^{15}\right)_\infty^3(q;q)_\infty^2\left(q^5;q^5\right)_\infty\\
&\quad=3q\frac{f^3(-q^{15})f^2(-q)}{f(-q^5)}.
\end{aligned}$$

The last line establishes (4.1), and hence concludes the proof. □

Second proof of (i). For this proof, we fix

$$\omega=e^{2\pi i/3}.$$

Now, applying (3.15) with $a=-q^2$, $b=-q^3$, and again with $a=-q$, $b=-q^4$, we deduce after rearranging that

$$f\left(-q^6,-q^9\right)=\frac{f(-q^{15})}{f^3(-q^5)}f\left(-q^2,-q^3\right)f\left(-\omega q^2,-\omega^2q^3\right)f\left(-\omega^2q^2,-\omega q^3\right)\tag{4.4}$$

and

$$f\left(-q^3,-q^{12}\right)=\frac{f(-q^{15})}{f^3(-q^5)}f\left(-q,-q^4\right)f\left(-\omega q,-\omega^2q^4\right)f\left(-\omega^2q,-\omega q^4\right).\tag{4.5}$$

As in the first proof of (i), we prove the equivalent formulation (4.1). Substituting (4.4) and (4.5) into (4.1), we see that, in order to prove (4.1), it is sufficient to prove that

$$\begin{aligned}
&\frac{f(-q^{15})}{f^3(-q^5)}\big(f^3(-q^2,-q^3)f(-q,-q^4)f(-\omega q,-\omega^2 q^4)f(-\omega^2 q,-\omega q^4)\\
&\quad - f^3(-q,-q^4)f(-q^2,-q^3)f(-\omega q^2,-\omega^2 q^3)f(-\omega^2 q^2,-\omega q^3)\big)\\
&= 3q\frac{f^3(-q^{15})f^2(-q)}{f(-q^5)}.
\end{aligned} \tag{4.6}$$

By (2.6), it is easy to verify that

$$f(-q,-q^4)f(-q^2,-q^3) = f(-q)f(-q^5). \tag{4.7}$$

Applying (4.7) twice in (4.6) and simplifying, we deduce that (4.1) is equivalent to

$$\begin{aligned}
&f^2(-q^2,-q^3)f(-\omega q,-\omega^2 q^4)f(-\omega^2 q,-\omega q^4)\\
&\quad - f^2(-q,-q^4)f(-\omega q^2,-\omega^2 q^3)f(-\omega^2 q^2,-\omega q^3) = 3qf^2(-q^{15})f(-q^5)f(-q).
\end{aligned} \tag{4.8}$$

Applying Lemma 2.3, first with $a=-\omega q$, $b=-\omega^2 q^4$, $c=-q^2$, $d=-q^3$; secondly with $a=-\omega^2 q$, $b=-\omega q^4$, $c=-q^2$, $d=-q^3$; thirdly with $a=-q$, $b=-q^4$, $c=-\omega q^2$, $d=-\omega^2 q^3$; and fourthly with $a=-q$, $b=-q^4$, $c=-\omega^2 q^2$, $d=-\omega q^3$, we deduce, respectively, that

$$\begin{aligned}
f(-\omega q,-\omega^2 q^4)f(-q^2,-q^3) &= f(\omega q^3,\omega^2 q^7)f(\omega q^4,\omega^2 q^6)\\
&\quad - \omega q f(\omega^2 q^2,\omega q^8)f(\omega^2 q,\omega q^9),
\end{aligned} \tag{4.9}$$

$$\begin{aligned}
f(-\omega^2 q,-\omega q^4)f(-q^2,-q^3) &= f(\omega^2 q^3,\omega q^7)f(\omega^2 q^4,\omega q^6)\\
&\quad - \omega^2 q f(\omega q^2,\omega^2 q^8)f(\omega q,\omega^2 q^9),
\end{aligned} \tag{4.10}$$

$$\begin{aligned}
f(-q,-q^4)f(-\omega q^2,-\omega^2 q^3) &= f(\omega q^3,\omega^2 q^7)f(\omega^2 q^4,\omega q^6)\\
&\quad - q f(\omega^2 q^2,\omega q^8)f(\omega q,\omega^2 q^9),
\end{aligned} \tag{4.11}$$

and

$$\begin{aligned}
f(-q,-q^4)f(-\omega^2 q^2,-\omega q^3) &= f(\omega^2 q^3,\omega q^7)f(\omega q^4,\omega^2 q^6)\\
&\quad - q f(\omega q^2,\omega^2 q^8)f(\omega^2 q,\omega q^9).
\end{aligned} \tag{4.12}$$

Substituting (4.9)–(4.12) into (4.8), we see that (4.1) is equivalent to

$$\begin{aligned}
&\big(f(\omega q^3,\omega^2 q^7)f(\omega q^4,\omega^2 q^6) - \omega q f(\omega^2 q^2,\omega q^8)f(\omega^2 q,\omega q^9)\big)\\
&\quad \times \big(f(\omega^2 q^3,\omega q^7)f(\omega^2 q^4,\omega q^6) - \omega^2 q f(\omega q^2,\omega^2 q^8)f(\omega q,\omega^2 q^9)\big)\\
&\quad - \big(f(\omega q^3,\omega^2 q^7)f(\omega^2 q^4,\omega q^6) - q f(\omega^2 q^2,\omega q^8)f(\omega q,\omega^2 q^9)\big)\\
&\quad \times \big(f(\omega^2 q^3,\omega q^7)f(\omega q^4,\omega^2 q^6) - q f(\omega q^2,\omega^2 q^8)f(\omega^2 q,\omega q^9)\big)\\
&= 3qf^2(-q^{15})f(-q^5)f(-q).
\end{aligned} \tag{4.13}$$

Expanding out the left-hand side of (4.13) and performing the obvious simplifications, we deduce that, in order to prove (4.1), it suffices to prove that

$$
\begin{aligned}
& f\left(\omega q^3, \omega^2 q^7\right) f\left(\omega^2 q^4, \omega q^6\right) f\left(\omega q^2, \omega^2 q^8\right) f\left(\omega^2 q, \omega q^9\right) \\
&\quad + f\left(\omega^2 q^3, \omega q^7\right) f\left(\omega q^4, \omega^2 q^6\right) f\left(\omega^2 q^2, \omega q^8\right) f\left(\omega q, \omega^2 q^9\right) \\
&\quad - \omega^2 f\left(\omega q^3, \omega^2 q^7\right) f\left(\omega q^4, \omega^2 q^6\right) f\left(\omega q^2, \omega^2 q^8\right) f\left(\omega q, \omega^2 q^9\right) \\
&\quad - \omega f\left(\omega^2 q^3, \omega q^7\right) f\left(\omega^2 q^4, \omega q^6\right) f\left(\omega^2 q^2, \omega q^8\right) f\left(\omega^2 q, \omega q^9\right) \\
&= 3 f^2\left(-q^{15}\right) f\left(-q^5\right) f(-q). \qquad (4.14)
\end{aligned}
$$

Next, we employ (2.11) with $n = 3$ along with one application of (2.5) to deduce that

$$f(a, b) = f\left(a^3 b^6, a^6 b^3\right) + a f\left(b^3, a^9 b^6\right) + b f\left(a^3, a^6 b^9\right). \qquad (4.15)$$

Choosing $a = \omega q^3$ and $b = \omega^2 q^7$ in (4.15), we find that

$$f\left(\omega q^3, \omega^2 q^7\right) = f\left(q^{51}, q^{39}\right) + \omega q^3 f\left(q^{21}, q^{69}\right) + \omega^2 q^7 f\left(q^9, q^{81}\right). \qquad (4.16)$$

Replacing ω by ω^2 in (4.16), we find that

$$f\left(\omega^2 q^3, \omega q^7\right) = f\left(q^{51}, q^{39}\right) + \omega^2 q^3 f\left(q^{21}, q^{69}\right) + \omega q^7 f\left(q^9, q^{81}\right). \qquad (4.17)$$

Similarly, we deduce the relations

$$f\left(\omega q^4, \omega^2 q^6\right) = f\left(q^{48}, q^{42}\right) + \omega q^4 f\left(q^{18}, q^{72}\right) + \omega^2 q^6 f\left(q^{12}, q^{78}\right), \qquad (4.18)$$

$$f\left(\omega^2 q^4, \omega q^6\right) = f\left(q^{48}, q^{42}\right) + \omega^2 q^4 f\left(q^{18}, q^{72}\right) + \omega q^6 f\left(q^{12}, q^{78}\right), \qquad (4.19)$$

$$f\left(\omega q^2, \omega^2 q^8\right) = f\left(q^{54}, q^{36}\right) + \omega q^2 f\left(q^{24}, q^{66}\right) + \omega^2 q^8 f\left(q^6, q^{84}\right), \qquad (4.20)$$

$$f\left(\omega^2 q^2, \omega q^8\right) = f\left(q^{54}, q^{36}\right) + \omega^2 q^2 f\left(q^{24}, q^{66}\right) + \omega q^8 f\left(q^6, q^{84}\right), \qquad (4.21)$$

$$f\left(\omega q, \omega^2 q^9\right) = f\left(q^{57}, q^{33}\right) + \omega q f\left(q^{27}, q^{63}\right) + \omega^2 q^9 f\left(q^3, q^{87}\right), \qquad (4.22)$$

and

$$f\left(\omega^2 q, \omega q^9\right) = f\left(q^{57}, q^{33}\right) + \omega^2 q f\left(q^{27}, q^{63}\right) + \omega q^9 f\left(q^3, q^{87}\right). \qquad (4.23)$$

Substituting (4.16)–(4.23) into the left-hand side of (4.14), expanding the resulting expression, simplifying, and then factoring we find that (4.1) is equivalent to

$$
\begin{aligned}
& 3\left[q^{15} f\left(q^{12}, q^{78}\right) f\left(q^3, q^{87}\right) - q^{13} f\left(q^{18}, q^{72}\right) f\left(q^3, q^{87}\right) + q f\left(q^{42}, q^{48}\right) f\left(q^{27}, q^{63}\right)\right. \\
&\quad \left. - f\left(q^{42}, q^{48}\right) f\left(q^{33}, q^{57}\right) - q^7 f\left(q^{12}, q^{78}\right) f\left(q^{27}, q^{63}\right) + q^4 f\left(q^{18}, q^{72}\right) f\left(q^{33}, q^{57}\right)\right] \\
&\quad \times \left[q^{15} f\left(q^9, q^{81}\right) f\left(q^6, q^{84}\right) - q^{11} f\left(q^{21}, q^{69}\right) f\left(q^6, q^{84}\right) - q^9 f\left(q^9, q^{81}\right) f\left(q^{24}, q^{26}\right)\right. \\
&\quad \left. + q^3 f\left(q^{36}, q^{54}\right) f\left(q^{21}, q^{69}\right) + q^2 f\left(q^{39}, q^{51}\right) f\left(q^{24}, q^{66}\right) - f\left(q^{36}, q^{54}\right) f\left(q^{39}, q^{51}\right)\right] \\
&= 3 f^2\left(-q^{15}\right) f\left(-q^5\right) f(-q). \qquad (4.24)
\end{aligned}
$$

Apply Theorem 3.3 to the two bracketed expressions above. Accordingly, we deduce that (4.1) is equivalent to

$$
\begin{aligned}
&3\big[-f\big(-q,-q^4\big)f\big(-q^{15},-q^{30}\big)\big]\times\big[-f\big(-q^2,-q^3\big)f\big(-q^{15},-q^{30}\big)\big]\\
&\quad=3f^2\big(-q^{15}\big)f\big(-q^5\big)f(-q).
\end{aligned}
\tag{4.25}
$$

Applying (2.9) and (4.7), we readily deduce the truth of (4.25). This then proves (4.1), and hence completes our second proof of (i). □

Proof of (ii). Applying (2.14), we rewrite (ii) in the equivalent form

$$
f^3\big(-q^6,-q^9\big)f\big(-q^2,-q^3\big)+q^2f^3\big(-q^3,-q^{12}\big)f\big(-q,-q^4\big)=\frac{f^3(-q^5)f^2(-q^3)}{f(-q^{15})}.
\tag{4.26}
$$

We prove (4.26). By (2.6), we verify that

$$
f\big(-q^2,-q^3\big)=f\big(-q^2,-q^{13}\big)f\big(-q^3,-q^{12}\big)f\big(-q^7,-q^8\big)\frac{f(-q^5)}{f^3(-q^{15})}
\tag{4.27}
$$

and

$$
f\big(-q,-q^4\big)=f\big(-q,-q^{14}\big)f\big(-q^6,-q^9\big)f\big(-q^4,-q^{11}\big)\frac{f(-q^5)}{f^3(-q^{15})}.
\tag{4.28}
$$

Identities (4.27)–(4.28) are also recorded explicitly in Ramanujan's notebooks [5, p. 222, Entry 2(i)]. Employing these, we find that (4.26) is equivalent to

$$
\begin{aligned}
&f^3\big(-q^6,-q^9\big)f\big(-q^2,-q^{13}\big)f\big(-q^3,-q^{12}\big)f\big(-q^7,-q^8\big)\frac{f(-q^5)}{f^3(-q^{15})}\\
&\qquad+q^2f^3\big(-q^3,-q^{12}\big)f\big(-q,-q^{14}\big)f\big(-q^6,-q^9\big)f\big(-q^4,-q^{11}\big)\frac{f(-q^5)}{f^3(-q^{15})}\\
&\quad=\frac{f^3(-q^5)f^2(-q^3)}{f(-q^{15})}.
\end{aligned}
\tag{4.29}
$$

Replacing q by q^3 in (4.7), we see that

$$
f\big(-q^3,-q^{12}\big)f\big(-q^6,-q^9\big)=f\big(-q^3\big)f\big(-q^{15}\big).
\tag{4.30}
$$

Utilizing (4.30) twice in (4.29) and simplifying, we deduce that (4.26) is equivalent to

$$
\begin{aligned}
&f^2\big(-q^6,-q^9\big)f\big(-q^2,-q^{13}\big)f\big(-q^7,-q^8\big)\\
&\quad+q^2f^2\big(-q^3,-q^{12}\big)f\big(-q,-q^{14}\big)f\big(-q^4,-q^{11}\big)=f\big(-q^{15}\big)f^2\big(-q^5\big)f\big(-q^3\big).
\end{aligned}
\tag{4.31}
$$

Next, we apply Lemma 2.3 four times, first with $a=-q^2$, $b=-q^{13}$, $c=-q^6$, $d=-q^9$; secondly with $a=-q^7$, $b=-q^8$, $c=-q^6$, $d=-q^9$; thirdly with $a=-q$, $b=-q^{14}$, $c=-q^3$, $d=-q^{12}$; and fourthly with $a=-q^4$, $b=-q^{11}$, $c=-q^3$, $d=-q^{12}$, along with (2.5) in the second and fourth cases. This yields the identities

$$f\left(-q^2,-q^{13}\right)f\left(-q^6,-q^9\right)=f\left(q^8,q^{22}\right)f\left(q^{11},q^{19}\right)-q^2f\left(q^7,q^{23}\right)f\left(q^4,q^{26}\right), \tag{4.32}$$

$$f\left(-q^7,-q^8\right)f\left(-q^6,-q^9\right)=f\left(q^{13},q^{17}\right)f\left(q^{14},q^{16}\right)-q^6f\left(q^2,q^{28}\right)f\left(q,q^{29}\right), \tag{4.33}$$

$$f\left(-q,-q^{14}\right)f\left(-q^3,-q^{12}\right)=f\left(q^4,q^{26}\right)f\left(q^{13},q^{17}\right)-qf\left(q^{11},q^{19}\right)f\left(q^2,q^{28}\right), \tag{4.34}$$

$$f\left(-q^4,-q^{11}\right)f\left(-q^3,-q^{12}\right)=f\left(q^7,q^{23}\right)f\left(q^{14},q^{16}\right)-q^3f\left(q^8,q^{22}\right)f\left(q,q^{29}\right). \tag{4.35}$$

Substituting (4.32)–(4.35) into (4.31), we see that (4.26) is equivalent to

$$\begin{aligned}&\left[f\left(q^8,q^{22}\right)f\left(q^{11},q^{19}\right)-q^2f\left(q^7,q^{23}\right)f\left(q^4,q^{26}\right)\right]\\&\quad\times\left[f\left(q^{13},q^{17}\right)f\left(q^{14},q^{16}\right)-q^6f\left(q^2,q^{28}\right)f\left(q,q^{29}\right)\right]\\&\quad+q^2\left[f\left(q^4,q^{26}\right)f\left(q^{13},q^{17}\right)-qf\left(q^{11},q^{19}\right)f\left(q^2,q^{28}\right)\right]\\&\quad\times\left[f\left(q^7,q^{23}\right)f\left(q^{14},q^{16}\right)-q^3f\left(q^8,q^{22}\right)f\left(q,q^{29}\right)\right]=f\left(-q^{15}\right)f^2\left(-q^5\right)f\left(-q^3\right).\end{aligned} \tag{4.36}$$

Expanding out the left-hand side of (4.36), performing the obvious cancellations, and then factoring the resulting expression, we find that (4.26) is equivalent to

$$\begin{aligned}&\left[f\left(q^{11},q^{19}\right)f\left(q^{14},q^{16}\right)-q^5f\left(q^4,q^{26}\right)f\left(q,q^{29}\right)\right]\\&\quad\times\left[f\left(q^8,q^{22}\right)f\left(q^{13},q^{17}\right)-q^3f\left(q^2,q^{28}\right)f\left(q^7,q^{23}\right)\right]=f\left(-q^{15}\right)f^2\left(-q^5\right)f\left(-q^3\right).\end{aligned} \tag{4.37}$$

Next, we apply Lemma 2.3 twice more, first with $a=-q^5$, $b=-q^{10}$, $c=-q^6$, $d=-q^9$; and secondly with $a=-q^3$, $b=-q^{12}$, $c=-q^5$, $d=-q^{10}$. Accordingly, we deduce that

$$f\left(-q^6,-q^9\right)f\left(-q^5,-q^{10}\right)=f\left(q^{11},q^{19}\right)f\left(q^{14},q^{16}\right)-q^5f\left(q^4,q^{26}\right)f\left(q,q^{29}\right), \tag{4.38}$$

$$f\left(-q^3,-q^{12}\right)f\left(-q^5,-q^{10}\right)=f\left(q^8,q^{22}\right)f\left(q^{13},q^{17}\right)-q^3f\left(q^7,q^{23}\right)f\left(q^2,q^{28}\right). \tag{4.39}$$

Substituting (4.38)–(4.39) into (4.37), we see that, in order to prove (4.26), it suffices to prove that

$$\begin{aligned}&\left[f\left(-q^6,-q^9\right)f\left(-q^5,-q^{10}\right)\right]\times\left[f\left(-q^3,-q^{12}\right)f\left(-q^5,-q^{10}\right)\right]\\&\quad=f\left(-q^{15}\right)f^2\left(-q^5\right)f\left(-q^3\right).\end{aligned} \tag{4.40}$$

Employing (2.9) and (4.30), we readily deduce the truth of (4.40). This then completes the proof. □

5. Applications to the Rogers–Ramanujan continued fraction

We begin by offering new identities for the Rogers–Ramanujan continued fraction.

Theorem 5.1. *Let $u=R(q)$ and $v=R(q^3)$. Then*

(i) $$v-u^3=3q^{8/5}\frac{f^3(-q^{15})}{f(-q)f(-q^3)f(-q^5)G^3(q)G(q^3)},$$

(ii) $$1+uv^3=\frac{f^3(-q^5)}{f(-q)f(-q^3)f(-q^{15})G(q)G^3(q^3)},$$

(iii) $$\frac{v-u^3}{1+uv^3} = 3q^2\frac{f^5(-q^{15})f(-q)}{f^5(-q^5)f(-q^3)}\cdot\frac{u}{v},$$

(iv) $$\frac{v}{u^3}+\frac{u^3}{v} = 9q^2\frac{f^5(-q^{15})f(-q)}{f^5(-q^5)f(-q^3)}+2 = 3\frac{v-u^3}{1+uv^3}\cdot\frac{v}{u}+2,$$

(v) $$\frac{1}{uv^3}+uv^3 = \frac{f^5(-q^5)f(-q^3)}{q^2f^5(-q^{15})f(-q)}-2 = 3\frac{1+uv^3}{v-u^3}\cdot\frac{u}{v}-2.$$

Proof of (i), (ii). Multiply the identity in Theorem 4.1(i) by $q^{3/5}G^{-3}(q)G^{-1}(q^3)$ and apply (1.5). Accordingly, we deduce (i). Similarly, multiplying the identity in Theorem 4.1(ii) by $G^{-1}(q)G^{-3}(q^3)$ and applying (1.5), we deduce (ii). □

Proof of (iii). Divide the identities in (i) and (ii). The left-hand side of (iii) is immediate. Observe that, by (2.15), and (1.5), the right-hand side of the resulting identity is equal to

$$\begin{aligned}
3q^{8/5}\frac{f^4(-q^{15})}{f^4(-q^5)}\frac{G^2(q^3)}{G^2(q)} &= 3q^{8/5}\frac{f^4(-q^{15})}{f^4(-q^5)}\cdot\frac{f(-q)}{f(-q^5)}\frac{f(-q^{15})}{f(-q^3)}\frac{G(q)H(q)}{G(q^3)H(q^3)}\cdot\frac{G^2(q^3)}{G^2(q)}\\
&= 3q^2\frac{f^5(-q^{15})f(-q)}{f^5(-q^5)f(-q^3)}\cdot\frac{G(q^3)}{q^{3/5}H(q^3)}\cdot\frac{q^{1/5}H(q)}{G(q)}\\
&= 3q^2\frac{f^5(-q^{15})f(-q)}{f^5(-q^5)f(-q^3)}\cdot\frac{u}{v}.
\end{aligned}$$

This completes the proof. □

Proof of (iv), (v). Square the identity in (i), divide the resulting equation by u^3v, and rearrange the result to find that

$$\frac{v}{u^3}+\frac{u^3}{v} = 9q^{16/5}\frac{f^6(-q^{15})}{f^2(-q)f^2(-q^3)f^2(-q^5)G^6(q)G^2(q^3)u^3v}+2. \tag{5.1}$$

With the help of (1.5) and (2.15), we deduce that

$$\begin{aligned}
&9q^{16/5}\frac{f^6(-q^{15})}{f^2(-q)f^2(-q^3)f^2(-q^5)G^6(q)G^2(q^3)u^3v}\\
&= 9q^{16/5}\frac{f^6(-q^{15})}{f^2(-q)f^2(-q^3)f^2(-q^5)G^6(q)G^2(q^3)}\cdot\frac{G^3(q)G(q^3)}{q^{6/5}H^3(q)H(q^3)}\\
&= 9q^2\frac{f^6(-q^{15})}{f^2(-q)f^2(-q^3)f^2(-q^5)}\cdot\frac{1}{G^3(q)H^3(q)G(q^3)H(q^3)}\\
&= 9q^2\frac{f^6(-q^{15})}{f^2(-q)f^2(-q^3)f^2(-q^5)}\cdot\frac{f^3(-q)f(-q^3)}{f^3(-q^5)f(-q^{15})}\\
&= 9q^2\frac{f^5(-q^{15})f(-q)}{f^5(-q^5)f(-q^3)}.
\end{aligned} \tag{5.2}$$

Combining (5.1) with (5.2), we deduce the first equality in (iv). To prove the second equality, observe that, by (iii),

$$3q^2\frac{f^5(-q^{15})f(-q)}{f^5(-q^5)f(-q^3)}=\frac{v-u^3}{1+uv^3}\cdot\frac{v}{u}. \tag{5.3}$$

Substituting (5.3) into the first equality of (iv), we deduce the second equality.

Analogously, we deduce (v) from (ii). This completes the proof. □

In his notebooks, Ramanujan recorded the following exquisite modular equation of degree three [29, p. 321], [7, p. 17]; see also Ramanujan's Lost Notebook [30, p. 365], [2, p. 92].

Theorem 5.2. *Let*

$$u:=\frac{q^{1/5}}{1}\,\frac{}{+}\,\frac{q}{1}\,\frac{}{+}\,\frac{q^2}{1}\,\frac{}{+}\,\frac{q^3}{1}\,\frac{}{+}\cdots$$

and

$$v:=\frac{q^{3/5}}{1}\,\frac{}{+}\,\frac{q^3}{1}\,\frac{}{+}\,\frac{q^6}{1}\,\frac{}{+}\,\frac{q^9}{1}\,\frac{}{+}\cdots.$$

Then

$$\left(v-u^3\right)\left(1+uv^3\right)=3u^2v^2.$$

Only two proofs of Theorem 5.2 are known in the literature. The first is due to Rogers [33, p. 392, Eq. (6.2)], who uses the classical theory of theta functions. The second is due to Yi [41], who utilizes eta-function identities. We offer a new proof, based on Theorem 5.1.

Proof. By Theorem 5.1(i), (ii), it suffices to prove that

$$\frac{3q^{8/5}f^3(-q^{15})}{f(-q)f(-q^3)f(-q^5)G^3(q)G(q^3)}\cdot\frac{f^3(-q^5)}{f(-q)f(-q^3)f(-q^{15})G(q)G^3(q^3)}=3u^2v^2. \tag{5.4}$$

With the use of (2.15) and (1.5), we readily deduce the truth of (5.4), and thus complete the proof. □

Remark. We remark that Theorem 5.1 is the first to provide closed-form product representations for the factors appearing on the left-hand side of Ramanujan's modular equation in Theorem 5.2.

Next, we discuss further results of Ramanujan and corollaries of our work. In his notebooks, Ramanujan recorded the following entry connected with u and v.

Lemma 5.3. *(See [7, Entry 4, p. 17].) Let u and v be defined as above. Let*

$$m:=q^{2/5}\frac{f(-q^4,-q^{11})}{f(-q^7,-q^8)}v$$

and

$$n:=q^{2/5}\frac{f(-q,-q^{14})}{f(-q^2,-q^{13})}v.$$

Then

$$m-n=mn=\frac{m^2}{1+m}=\frac{n^2}{1-n}=uv^3.$$

See [7, pp. 17–18] for a proof. From Lemma 5.3 and Theorem 5.1(ii), we easily deduce the following new companion identities.

Corollary 5.4. *Let*

$$T := \frac{f^3(-q^5)}{f(-q)f(-q^3)f(-q^{15})G(q)G^3(q^3)}.$$

Then

$$m-n+1 = mn+1 = \frac{m^2+m+1}{m+1} = \frac{n^2-n+1}{1-n} = 1+uv^3 = T,$$
$$m^2+m+1 = \frac{Tm^2}{uv^3},$$

and

$$n^2-n+1 = \frac{Tn^2}{uv^3}.$$

The following lemma is stated on p. 205 of Ramanujan's Lost Notebook, and has been proved by J. Sohn [35].

Lemma 5.5. *(See [2, p. 45, Entry 1.10.2], [35].) Let $\omega = \exp(2\pi i/3)$, $u = R(q)$, and $v = R(q^3)$. If*

$$R := \frac{f^2(-q^3)}{qf^2(-q^{15})} = \left(\frac{1}{v^5} - 11 - v^5\right)^{1/3},$$

then

$$4u = -\frac{1}{v^3} - \sqrt{\frac{1}{v^6} - \frac{8+4R}{v}} + \sqrt{\frac{1}{v^6} - \frac{8+4R\omega}{v}} + \sqrt{\frac{1}{v^6} - \frac{8+4R\omega^2}{v}}. \tag{5.5}$$

If

$$R := \frac{f^2(-q)}{q^{1/3}f^2(-q^5)} = \left(\frac{1}{u^5} - 11 - u^5\right)^{1/3},$$

then

$$4v = u^3 - \sqrt{u^6 + u(8+4R)} + \sqrt{u^6 + (8+4R\omega)} + \sqrt{u^6 + u\big(8+4R\omega^2\big)}. \tag{5.6}$$

From Lemma 5.5 and Theorem 5.1, we obtain the following corollary.

Corollary 5.6. *Let $\omega = \exp(2\pi i/3)$, $u = R(q)$, and $v = R(q^3)$. If*

$$R := \frac{f^2(-q^3)}{qf^2(-q^{15})} = \left(\frac{1}{v^5} - 11 - v^5\right)^{1/3},$$

then

$$\frac{4f^3(-q^5)}{f(-q)f(-q^3)f(-q^{15})} = G(q)G^3(q^3)\left(3 - v^3\sqrt{\frac{1}{v^6} - \frac{8+4R}{v}} + v^3\sqrt{\frac{1}{v^6} - \frac{8+4R\omega}{v}} + v^3\sqrt{\frac{1}{v^6} - \frac{8+4R\omega^2}{v}}\right). \tag{5.7}$$

If

$$R := \frac{f^2(-q)}{q^{1/3}f^2(-q^5)} = \left(\frac{1}{u^5} - 11 - u^5\right)^{1/3},$$

then

$$\frac{12q^{8/5}f^3(-q^{15})}{f(-q)f(-q^3)f(-q^5)} = G^3(q)G(q^3)\big(-3u^3 - \sqrt{u^6+u(8+4R)} + \sqrt{u^6+(8+4R\omega)} + \sqrt{u^6+u\big(8+4R\omega^2\big)}\big). \tag{5.8}$$

6. Identities with cubes of the Ramanujan–Göllnitz–Gordon functions

In this section, we derive analogues of Theorem 4.1 and Corollary 4.2 for the Ramanujan–Göllnitz–Gordon functions. Our methods are similar to those used in Section 4. We record our results as Theorem 6.1.

Theorem 6.1. *We have*

(i) $$S^3(q)T\big(q^3\big) - T^3(q)S\big(q^3\big) = 3q\frac{f^3(-q^{24})}{f^3(-q^8)}\cdot\frac{\psi(-q^2)\phi(-q^4)}{\psi(-q)\psi(-q^3)}S(q)T(q),$$

(ii) $$S^3\big(q^3\big)S(q) + q^5T^3\big(q^3\big)T(q) = \frac{f^3(-q^8)}{f^3(-q^{24})}\cdot\frac{\psi(-q^6)\phi(-q^{12})}{\psi(-q)\psi(-q^3)}S\big(q^3\big)T\big(q^3\big),$$

(iii) $$\frac{S^3(q)T(q^3) - T^3(q)S(q^3)}{S^3(q^3)S(q) + q^5T^3(q^3)T(q)} = 3q\frac{f^4(-q^{24})}{f^4(-q^8)}\cdot\frac{\chi(q)}{\chi(q^3)}.$$

We offer two proofs of (i).

First proof of (i). Recall the notation of Theorem 3.5. By (2.16) and Theorem 3.5,

$$\begin{aligned}\frac{T(q^3)}{S(q^3)} - \frac{T^3(q)}{S^3(q)} &= \frac{f(-q^3,-q^{21})}{f(-q^9,-q^{15})} - \left(\frac{f(-q,-q^7)}{f(-q^3,-q^5)}\right)^3 = \frac{A^3-B^3}{C^3+D^3} - \left(\frac{A-B}{C+D}\right)^3 \\ &= \frac{(A^3-B^3)(C+D)^3 - (A-B)^3(C^3+D^3)}{(C^3+D^3)(C+D)^3} \\ &= \frac{3(A-B)(C+D)(AD+BC)(AC+BD)}{(C^3+D^3)(C+D)^3} \\ &= 3q\frac{f(q)f(-q^3,-q^{21})f^2(-q,-q^7)f(-q^4)f^2(-q^{24})\psi(q^3)}{f(-q^3,-q^5)f(-q^2,-q^6)f(-q^6,-q^{18})f^4(-q^8)f(-q^{12})}.\end{aligned} \tag{6.1}$$

Now, by (1.4) and (2.6),

$$S(q)=\frac{f^2(-q^8)}{f(-q,-q^7)f(-q^4)} \quad \text{and} \quad T(q)=\frac{f^2(-q^8)}{f(-q^3,-q^5)f(-q^4)}. \tag{6.2}$$

Employing (2.8), Lemma 2.1, and (6.2), we find that the last expression in (6.1) is equal to

$$\begin{aligned}
&3q\frac{f(q)f(-q^3,-q^{21})f^2(-q,-q^7)f(-q^4)f^2(-q^{24})\psi(q^3)}{f(-q^3,-q^5)\psi(-q^2)\psi(-q^6)f^4(-q^8)f(-q^{12})}\\
&\quad=3q\frac{f(-q^3,-q^{21})f^2(-q,-q^7)f(-q^4)f^2(-q^{24})}{f(-q^3,-q^5)f^4(-q^8)f(-q^{12})}\cdot\frac{f^3(-q^2)}{f(-q)f(-q^4)}\cdot\frac{f^2(-q^6)}{f(-q^3)}\\
&\qquad\times\frac{f(-q^4)}{f(-q^2)f(-q^8)}\cdot\frac{f(-q^{12})}{f(-q^6)f(-q^{24})}\\
&\quad=3q\left[\frac{f(-q^3,-q^{21})f(-q^{12})}{f^2(-q^{24})}\right]\cdot\left[\frac{f^2(-q,-q^7)f^2(-q^4)}{f^4(-q^8)}\right]\cdot\left[\frac{f^2(-q^8)}{f(-q^3,-q^5)f(-q^4)}\right]\\
&\qquad\times\left[\frac{f^3(-q^{24})}{f^3(-q^8)}\right]\cdot\left[\frac{f(-q^2)f(-q^8)}{f(-q^4)}\right]\cdot\left[\frac{f^2(-q^4)}{f(-q^8)}\right]\cdot\left[\frac{f(-q^2)}{f(-q)f(-q^4)}\right]\cdot\left[\frac{f(-q^6)}{f(-q^3)f(-q^{12})}\right]\\
&\quad=3q\frac{T(q)\psi(-q^2)\phi(-q^4)}{S(q^3)S^2(q)\psi(-q)\psi(-q^3)}\frac{f^3(-q^{24})}{f^3(-q^8)}.
\end{aligned}\tag{6.3}$$

Combining (6.1) and (6.3), we deduce that

$$\frac{T(q^3)}{S(q^3)}-\frac{T^3(q)}{S^3(q)}=3q\frac{f^3(-q^{24})}{f^3(-q^8)}\frac{\psi(-q^2)\phi(-q^4)}{S(q^3)S^2(q)\psi(-q)\psi(-q^3)}T(q). \tag{6.4}$$

Multiplying (6.4) by $S^3(q)S(q^3)$, we complete the proof. □

Second proof of (i). Applying (2.16), we rewrite (i) in the equivalent form

$$\begin{aligned}
&f^3(-q^3,-q^5)f(-q^3,-q^{21})-f^3(-q,-q^7)f(-q^9,-q^{15})\\
&\quad=3q\frac{f^3(-q^{24})}{f^3(-q^8)}\psi(-q^2)\phi(-q^4)f(-q^3,-q^5)f(-q,-q^7).
\end{aligned}\tag{6.5}$$

Set $\omega=e^{2\pi i/3}$. By (3.15), first with $a=-q$, $b=-q^7$; and second with $a=-q^3$, $b=-q^5$, we conclude that

$$f(-q^3,-q^{21})=\frac{f(-q^{24})}{f^3(-q^8)}f(-\omega q,-\omega^2q^7)f(-\omega^2q,-\omega q^7)f(-q,-q^7) \tag{6.6}$$

and

$$f(-q^9,-q^{15})=\frac{f(-q^{24})}{f^3(-q^8)}f(-\omega q^3,-\omega^2q^5)f(-\omega^2q^3,-\omega q^5)f(-q^3,-q^5). \tag{6.7}$$

Substituting (6.6) and (6.7) into (6.5) and simplifying, we deduce that (i) is equivalent to

$$f^2(-q^3,-q^5)f(-\omega q,-\omega^2q^7)f(-\omega^2q,-\omega q^7) \\ - f^2(-q,-q^7)f(-\omega q^3,-\omega^2q^5)f(-\omega^2q^3,-\omega q^5) = 3qf^2(-q^{24})\psi(-q^2)\phi(-q^4). \quad (6.8)$$

We prove (6.8). Applying Lemma 2.3, first with $a=-\omega q$, $b=-\omega^2q^7$, $c=-q^3$, $d=-q^5$; secondly with $a=-\omega^2q$, $b=-\omega q^7$, $c=-q^3$, $d=-q^5$; thirdly with $a=-q$, $b=-q^7$, $c=-\omega q^3$, $d=-\omega^2q^5$; and fourthly with $a=-q$, $b=-q^7$, $c=-\omega^2q^3$, $d=-\omega q^5$, we deduce, respectively, that

$$f(-\omega q,-\omega^2q^7)f(-q^3,-q^5) = f(\omega q^4,\omega^2q^{12})f(\omega q^6,\omega^2q^{10}) \\ - \omega qf(\omega^2q^4,\omega q^{12})f(\omega^2q^2,\omega q^{14}), \quad (6.9)$$

$$f(-\omega^2q,-\omega q^7)f(-q^3,-q^5) = f(\omega^2q^4,\omega q^{12})f(\omega^2q^6,\omega q^{10}) \\ - \omega^2qf(\omega q^4,\omega^2q^{12})f(\omega q^2,\omega^2q^{14}), \quad (6.10)$$

$$f(-q,-q^7)f(-\omega q^3,-\omega^2q^5) = f(\omega q^4,\omega^2q^{12})f(\omega^2q^6,\omega q^{10}) \\ - qf(\omega^2q^4,\omega q^{12})f(\omega q^2,\omega^2q^{14}), \quad (6.11)$$

and

$$f(-q,-q^7)f(-\omega^2q^3,-\omega q^5) = f(\omega^2q^4,\omega q^{12})f(\omega q^6,\omega^2q^{10}) \\ - qf(\omega q^4,\omega^2q^{12})f(\omega^2q^2,\omega q^{14}). \quad (6.12)$$

Substitute (6.9)–(6.12) into (6.8). Expanding out and then simplifying, we conclude that (6.8) is equivalent to

$$f^2(\omega q^4,\omega^2q^{12})f(\omega^2q^6,\omega q^{10})f(\omega^2q^2,\omega q^{14}) \\ + f^2(\omega^2q^4,\omega q^{12})f(\omega q^6,\omega^2q^{10})f(\omega q^2,\omega^2q^{14}) \\ - \omega^2f^2(\omega q^4,\omega^2q^{12})f(\omega q^6,\omega^2q^{10})f(\omega q^2,\omega^2q^{14}) \\ - \omega f^2(\omega^2q^4,\omega q^{12})f(\omega^2q^6,\omega q^{10})f(\omega^2q^2,\omega q^{14}) = 3f^2(-q^{24})\psi(-q^2)\phi(-q^4). \quad (6.13)$$

Next, we apply (4.15), first with $a=\omega q^4$, $b=\omega^2q^{12}$; secondly with $a=\omega q^6$, $b=\omega^2q^{10}$; and thirdly with $a=\omega q^2$, $b=\omega^2q^{14}$ in order to deduce, respectively, that

$$f(\omega q^4,\omega^2q^{12}) = f(q^{60},q^{84}) + \omega q^4f(q^{36},q^{108}) + \omega^2q^{12}f(q^{12},q^{132}), \quad (6.14)$$

$$f(\omega q^6,\omega^2q^{10}) = f(q^{66},q^{78}) + \omega q^6f(q^{30},q^{114}) + \omega^2q^{10}f(q^{18},q^{126}), \quad (6.15)$$

and

$$f(\omega q^2,\omega^2q^{14}) = f(q^{54},q^{90}) + \omega q^2f(q^{42},q^{102}) + \omega^2q^{14}f(q^6,q^{138}). \quad (6.16)$$

Replacing ω with ω^2, we obtain one further identity from each of (6.14)–(6.16). Substitute the six identities thus found into (6.13), expand out the resulting expressions, and simplify algebraically. We therefore find that (6.8) is equivalent to

$$\begin{aligned}
&\big[q^{20} f\big(q^{30}, q^{114}\big) f\big(q^{6}, q^{138}\big) - q^{24} f\big(q^{6}, q^{138}\big) f\big(q^{18}, q^{126}\big) \\
&\quad + q^{12} f\big(q^{18}, q^{126}\big) f\big(q^{42}, q^{102}\big) - q^{2} f\big(q^{42}, q^{102}\big) f\big(q^{66}, q^{78}\big) \\
&\quad + f\big(q^{66}, q^{78}\big) f\big(q^{54}, q^{90}\big) - q^{6} f\big(q^{54}, q^{90}\big) f\big(q^{30}, q^{114}\big)\big] \\
&\quad \times \big[f^2\big(q^{60}, q^{84}\big) - q^{24} f^2\big(q^{12}, q^{132}\big) + 2q^{16} f\big(q^{36}, q^{108}\big) f\big(q^{12}, q^{132}\big) \\
&\quad - 2q^{4} f\big(q^{36}, q^{108}\big) f\big(q^{60}, q^{84}\big)\big] = f^2\big(-q^{24}\big)\psi\big(-q^{2}\big)\phi\big(-q^{4}\big). \qquad (6.17)
\end{aligned}$$

With the help of Theorem 3.4(i), (iv), we readily deduce the truth of (6.17), and hence also of (6.8). This completes the proof. □

Proof of (ii). Employing (2.16), we rewrite (ii) in the equivalent form

$$\begin{aligned}
&f^3\big(-q^{9}, -q^{15}\big) f\big(-q^{3}, -q^{5}\big) + q^{5} f^3\big(-q^{3}, -q^{21}\big) f\big(-q, -q^{7}\big) \\
&\quad = \frac{f^3(-q^8)}{f^3(-q^{24})}\psi\big(-q^{6}\big)\phi\big(-q^{12}\big) f\big(-q^{9}, -q^{15}\big) f\big(-q^{3}, -q^{21}\big). \qquad (6.18)
\end{aligned}$$

We prove (6.18) by transforming the left-hand side into the right-hand side. Employing (2.6) and (2.9), we readily deduce that

$$f\big(-q^{3}, -q^{5}\big) = f\big(-q^{11}, -q^{13}\big) f\big(-q^{5}, -q^{19}\big) f\big(-q^{3}, -q^{21}\big)\frac{f(-q^8)}{f^3(-q^{24})} \qquad (6.19)$$

and

$$f\big(-q, -q^{7}\big) = f\big(-q, -q^{23}\big) f\big(-q^{9}, -q^{15}\big) f\big(-q^{7}, -q^{17}\big)\frac{f(-q^8)}{f^3(-q^{24})}. \qquad (6.20)$$

Applying (6.19) and (6.20), we find that the left-hand side of (6.18) is equal to

$$\begin{aligned}
&\frac{f(-q^8)}{f^3(-q^{24})} f\big(-q^{9}, -q^{15}\big) f\big(-q^{3}, -q^{21}\big) \\
&\quad \times \big\{f^2\big(-q^{9}, -q^{15}\big) f\big(-q^{11}, -q^{13}\big) f\big(-q^{5}, -q^{19}\big) \\
&\quad + q^{5} f^2\big(-q^{3}, -q^{21}\big) f\big(-q, -q^{23}\big) f\big(-q^{7}, -q^{17}\big)\big\}. \qquad (6.21)
\end{aligned}$$

Next, we apply Lemma 2.3 four times: first with $a=-q^9$, $b=-q^{15}$, $c=-q^{11}$, $d=-q^{13}$; secondly with $a=-q^9$, $b=-q^{15}$, $c=-q^5$, $d=-q^{19}$; thirdly with $a=-q^3$, $b=-q^{21}$, $c=-q$, $d=-q^{23}$; and fourthly with $a=-a^3$, $b=-q^{21}$, $c=-q^7$, and $d=-q^{17}$, in order to deduce, respectively, that

$$f\big(-q^{9}, -q^{15}\big) f\big(-q^{11}, -q^{13}\big) = f\big(q^{20}, q^{28}\big) f\big(q^{22}, q^{26}\big) - q^{9} f\big(q^{4}, q^{44}\big) f\big(q^{2}, q^{46}\big), \qquad (6.22)$$

$$f\big(-q^{9}, -q^{15}\big) f\big(-q^{5}, -q^{19}\big) = f\big(q^{14}, q^{34}\big) f\big(q^{20}, q^{28}\big) - q^{5} f\big(q^{4}, q^{44}\big) f\big(q^{10}, q^{38}\big), \qquad (6.23)$$

$$f\big(-q^{3}, -q^{21}\big) f\big(-q, -q^{23}\big) = f\big(q^{4}, q^{44}\big) f\big(q^{22}, q^{26}\big) - q f\big(q^{2}, q^{46}\big) f\big(q^{20}, q^{28}\big), \qquad (6.24)$$

and

$$f\big(-q^{3}, -q^{21}\big) f\big(-q^{7}, -q^{17}\big) = f\big(q^{10}, q^{38}\big) f\big(q^{20}, q^{28}\big) - q^{3} f\big(q^{4}, q^{44}\big) f\big(q^{14}, q^{34}\big). \qquad (6.25)$$

Here (2.5) was used to simplify (6.23) and (6.24). Substituting (6.22)–(6.25) into (6.21), we find that the left-hand side of (6.18) is equal to

$$\begin{aligned}
&\frac{f(-q^8)}{f^3(-q^{24})} f\left(-q^9,-q^{15}\right) f\left(-q^3,-q^{21}\right)\\
&\quad\times\left\{\left[f\left(q^{20},q^{28}\right) f\left(q^{22},q^{26}\right)-q^9 f\left(q^4,q^{44}\right) f\left(q^2,q^{46}\right)\right]\right.\\
&\quad\times\left[f\left(q^{14},q^{34}\right) f\left(q^{20},q^{28}\right)-q^5 f\left(q^4,q^{44}\right) f\left(q^{10},q^{38}\right)\right]\\
&\quad+q^5\left[f\left(q^4,q^{44}\right) f\left(q^{22},q^{26}\right)-q f\left(q^2,q^{46}\right) f\left(q^{20},q^{28}\right)\right]\\
&\quad\left.\times\left[f\left(q^{10},q^{38}\right) f\left(q^{20},q^{28}\right)-q^3 f\left(q^4,q^{44}\right) f\left(q^{14},q^{34}\right)\right]\right\},
\end{aligned}$$

which is algebraically equivalent to

$$\begin{aligned}
&\frac{f(-q^8)}{f^3(-q^{24})} f\left(-q^9,-q^{15}\right) f\left(-q^3,-q^{21}\right)\left\{\left[f^2\left(q^{20},q^{28}\right)-q^8 f^2\left(q^4,q^{44}\right)\right]\right.\\
&\quad\left.\times\left[f\left(q^{22},q^{26}\right) f\left(q^{14},q^{34}\right)-q^6 f\left(q^2,q^{46}\right) f\left(q^{10},q^{38}\right)\right]\right\}.
\end{aligned} \tag{6.26}$$

Next, we apply Lemma 2.3 two more times, first with $a=-q^8$, $b=-q^{16}$, $c=d=-q^{12}$; and secondly with $a=-q^8$, $b=-q^{16}$, $c=-q^6$, $d=-q^{18}$, along with one application of (2.5), in order to deduce that

$$f\left(-q^8,-q^{16}\right) f\left(-q^{12},-q^{12}\right)=f^2\left(q^{20},q^{28}\right)-q^8 f^2\left(q^4,q^{44}\right) \tag{6.27}$$

and

$$f\left(-q^8,-q^{16}\right) f\left(-q^6,-q^{18}\right)=f\left(q^{22},q^{26}\right) f\left(q^{14},q^{34}\right)-q^6 f\left(q^2,q^{46}\right) f\left(q^{10},q^{38}\right). \tag{6.28}$$

Substituting (6.27) and (6.28) into (6.26), we deduce that the left-hand side of (6.18) is equal to

$$\begin{aligned}
&\frac{f(-q^8)}{f^3(-q^{24})} f\left(-q^9,-q^{15}\right) f\left(-q^3,-q^{21}\right)\\
&\quad\times\left[f\left(-q^8,-q^{16}\right) f\left(-q^{12},-q^{12}\right)\right]\times\left[f\left(-q^8,-q^{16}\right) f\left(-q^6,-q^{18}\right)\right].
\end{aligned} \tag{6.29}$$

Employing (2.7)–(2.9) in (6.29), we conclude, finally, that the left-hand side of (6.18) is equal to

$$\frac{f^3(-q^8)}{f^3(-q^{24})} f\left(-q^9,-q^{15}\right) f\left(-q^3,-q^{21}\right)\psi\left(-q^6\right)\phi\left(-q^{12}\right). \tag{6.30}$$

This proves (6.18), and hence completes the proof. □

Proof of (iii). Divide (i) by (ii). On the right-hand side, apply (2.17) and Lemma 2.1 to complete the proof. □

7. Further relations for the Ramanujan–Göllnitz–Gordon functions

In this section, we develop four related identities for the Ramanujan–Göllnitz–Gordon functions, each of which connects these functions at the arguments q and q^3. New proofs for each relation are given, and one of our identities is new. In Section 8, we combine the results of this section with those in Section 6 in order to give applications to the Ramanujan–Göllnitz–Gordon continued fraction.

Theorem 7.1. *We have*

$$\text{(i)} \qquad S\left(q^3\right)T(q)+qS(q)T\left(q^3\right)=\frac{\phi(-q^4)\psi(-q^6)}{\psi(-q)\psi(-q^3)},$$

$$\text{(ii)} \qquad S\left(q^3\right)S(q)-q^2T\left(q^3\right)T(q)=\frac{\psi(-q^2)\phi(-q^{12})}{\psi(-q)\psi(-q^3)},$$

$$\text{(iii)} \qquad S\left(q^3\right)T(q)-qS(q)T\left(q^3\right)=\frac{f(-q)f(-q^{12})}{f(-q^3)f(-q^4)},$$

$$\text{(iv)} \qquad S\left(q^3\right)S(q)+q^2T\left(q^3\right)T(q)=\frac{f(-q^3)f(-q^4)}{f(-q)f(-q^{12})}.$$

Identities (i) and (iii) were first discovered by Robins [31], who utilized the theory of modular forms. Using techniques of Bressoud, Huang [25] proved both (iii) and (iv), and furthermore offered partition-theoretic interpretations of those two identities. Subsequently, Baruah, Bora, and Saikia [4] discovered new proofs of (iii) and (iv). Identity (ii) is new.

Proof of (i), (iii). Applying (2.16) and Lemma 2.1, we rewrite (i) and (iii), respectively, in the equivalent forms

$$f\left(-q^9,-q^{15}\right)f\left(-q,-q^7\right)+qf\left(-q^3,-q^5\right)f\left(-q^3,-q^{21}\right)=\phi\left(-q^4\right)\psi\left(-q^6\right), \tag{7.1}$$

$$f\left(-q^9,-q^{15}\right)f\left(-q,-q^7\right)-qf\left(-q^3,-q^5\right)f\left(-q^3,-q^{21}\right)=\phi(-q)\psi\left(q^6\right). \tag{7.2}$$

Adding, respectively subtracting, (7.1) and (7.2), we obtain

$$2f\left(-q^9,-q^{15}\right)f\left(-q,-q^7\right)=\phi\left(-q^4\right)\psi\left(-q^6\right)+\phi(-q)\psi\left(q^6\right), \tag{7.3}$$

$$2qf\left(-q^3,-q^5\right)f\left(-q^3,-q^{21}\right)=\phi\left(-q^4\right)\psi\left(-q^6\right)-\phi(-q)\psi\left(q^6\right). \tag{7.4}$$

It is clear that (7.3) and (7.4) together imply both (7.1) and (7.2), and hence also (i) and (iii). Thus, it suffices to prove (7.3) and (7.4). To that end, we employ (2.7), along with (2.11) with $a=b=-q$ and $n=2$, in order to see that

$$\phi(-q)=f(-q,-q)=f\left(q^4,q^4\right)-qf\left(1,q^8\right). \tag{7.5}$$

Alternatively, one can show (7.5) directly by considering the even-odd dissection of the sum in (2.7).

Combining (7.5) with (2.8), we find that

$$\phi(-q)\psi\left(q^6\right)=f\left(q^4,q^4\right)f\left(q^6,q^{18}\right)-qf\left(1,q^8\right)f\left(q^6,q^{18}\right). \tag{7.6}$$

Observe that, by (2.4),

$$f\left(-1,-q^8\right)f\left(-q^6,-q^{18}\right)=0. \tag{7.7}$$

Thus, by (2.7), (2.8), (7.6), and (7.7),

$$\phi(-q^4)\psi(-q^6)+\phi(-q)\psi(q^6)=f(-q^4,-q^4)f(-q^6,-q^{18})+f(q^4,q^4)f(q^6,q^{18}) \\ -qf(1,q^8)f(q^6,q^{18})+qf(-1,-q^8)f(-q^6,-q^{18}). \tag{7.8}$$

Next, apply Theorem 2.7 with the parameters $a=b=q^4$, $c=q^6$, $d=q^{18}$, $\alpha=1$, $\beta=3$, $\epsilon_1=\epsilon_2=0$, and $m=4$. We consequently find that

$$\begin{aligned} f(q^4,q^4)f(q^6,q^{18}) &= f(q^{10},q^{22})f(q^{42},q^{54})+q^6f(q^{-2},q^{34})f(q^{30},q^{66}) \\ &\quad +q^{36}f(q^{-26},q^{58})f(q^6,q^{90})+q^{90}f(q^{-50},q^{82})f(q^{-18},q^{114}) \\ &= f(q^{10},q^{22})f(q^{42},q^{54})+q^4f(q^2,q^{30})f(q^{30},q^{66}) \\ &\quad +q^{10}f(q^6,q^{26})f(q^6,q^{90})+q^4f(q^{14},q^{18})f(q^{18},q^{78}), \end{aligned} \tag{7.9}$$

where we applied (2.5) four times in the last equality. Applying Theorem 2.7 with the same set of parameters, except now with $\epsilon_1=\epsilon_2=1$, we similarly deduce that

$$f(-q^4,-q^4)f(-q^6,-q^{18})=f(q^{10},q^{22})f(q^{42},q^{54})-q^4f(q^2,q^{30})f(q^{30},q^{66}) \\ +q^{10}f(q^6,q^{26})f(q^6,q^{90})-q^4f(q^{14},q^{18})f(q^{18},q^{78}). \tag{7.10}$$

By a third application of Theorem 2.7, this time with the parameters $a=1$, $b=q^8$, $c=q^6$, $d=q^{18}$, $\alpha=1$, $\beta=3$, $\epsilon_1=\epsilon_2=0$, and $m=4$, we find that

$$\begin{aligned} f(1,q^8)f(q^6,q^{18}) &= f(q^{14},q^{18})f(q^{30},q^{66})+q^6f(q^{-6},q^{38})f(q^{42},q^{54}) \\ &\quad +q^{36}f(q^{-30},q^{62})f(q^{18},q^{78})+q^{90}f(q^{-54},q^{86})f(q^{-6},q^{102}) \\ &= f(q^{14},q^{18})f(q^{30},q^{66})+f(q^6,q^{26})f(q^{42},q^{54}) \\ &\quad +q^6f(q^2,q^{30})f(q^{18},q^{78})+q^8f(q^{10},q^{22})f(q^6,q^{90}). \end{aligned} \tag{7.11}$$

Employing Theorem 2.7 with the same set of parameters, but now with $\epsilon_1=\epsilon_2=1$, we deduce after simplifying that

$$f(-1,-q^8)f(-q^6,-q^{18})=f(q^{14},q^{18})f(q^{30},q^{66})-f(q^6,q^{26})f(q^{42},q^{54}) \\ +q^6f(q^2,q^{30})f(q^{18},q^{78})-q^8f(q^{10},q^{22})f(q^6,q^{90}). \tag{7.12}$$

Combining (7.8)–(7.12), we conclude that

$$\begin{aligned} &\phi(-q^4)\psi(-q^6)+\phi(-q)\psi(q^6) \\ &\quad =2f(q^{10},q^{22})f(q^{42},q^{54})-2qf(q^6,q^{26})f(q^{42},q^{54}) \\ &\qquad -2q^9f(q^{10},q^{22})f(q^6,q^{90})+2q^{10}f(q^6,q^{26})f(q^6,q^{90}) \\ &\quad =2\big(f(q^{42},q^{54})-q^9f(q^6,q^{90})\big)\big(f(q^{10},q^{22})-qf(q^6,q^{26})\big). \end{aligned} \tag{7.13}$$

By (2.11) with $a=-q$, $b=-q^7$, $n=2$, and with $a=-q^3$, $b=-q^5$, $n=2$, we deduce, respectively, that

$$f\left(-q,-q^7\right)=f\left(q^{10},q^{22}\right)-qf\left(q^6,q^{26}\right), \tag{7.14}$$

$$f\left(-q^3,-q^5\right)=f\left(q^{14},q^{18}\right)-q^3f\left(q^2,q^{30}\right). \tag{7.15}$$

Replacing q by q^3 in each of (7.14), (7.15), we find, respectively, that

$$f\left(-q^3,-q^{21}\right)=f\left(q^{30},q^{66}\right)-q^3f\left(q^{18},q^{78}\right), \tag{7.16}$$

$$f\left(-q^9,-q^{15}\right)=f\left(q^{42},q^{54}\right)-q^9f\left(q^6,q^{90}\right). \tag{7.17}$$

Thus, employing (7.17) and (7.14) in (7.13), we readily deduce the truth of (7.3).

Similarly, to prove (7.4), observe that, by (2.7), (2.8), (7.6), and (7.7),

$$\begin{aligned}\phi\left(-q^4\right)\psi\left(-q^6\right)-\phi(-q)\psi\left(q^6\right)&=f\left(-q^4,-q^4\right)f\left(-q^6,-q^{18}\right)-f\left(q^4,q^4\right)f\left(q^6,q^{18}\right)\\&\quad+qf\left(1,q^8\right)f\left(q^6,q^{18}\right)+qf\left(-1,-q^8\right)f\left(-q^6,-q^{18}\right).\end{aligned} \tag{7.18}$$

Applying (7.9)–(7.12), followed by (7.15) and (7.16), we conclude from (7.18) that

$$\begin{aligned}&\phi\left(-q^4\right)\psi\left(-q^6\right)-\phi(-q)\psi\left(q^6\right)\\&\quad=-2q^4f\left(q^2,q^{30}\right)f\left(q^{30},q^{66}\right)-2q^4f\left(q^{14},q^{18}\right)f\left(q^{18},q^{78}\right)\\&\qquad+2qf\left(q^{14},q^{18}\right)f\left(q^{30},q^{66}\right)+2q^7f\left(q^2,q^{30}\right)f\left(q^{18},q^{78}\right)\\&\quad=2q\left(f\left(q^{14},q^{18}\right)-q^3f\left(q^2,q^{30}\right)\right)\left(f\left(q^{30},q^{66}\right)-q^3f\left(q^{18},q^{78}\right)\right)\\&\quad=2qf\left(-q^3,-q^5\right)f\left(-q^3,-q^{21}\right).\end{aligned}$$

Thus (7.4) is proved, and we complete the proof of (i) and (iii).

Proof of (ii), (iv). The proof of (ii) and (iv) is analogous to the proof of (i) and (iii); we omit the details. □

8. Modular relations for the Ramanujan–Göllnitz–Gordon continued fraction

In this section, we apply the results of the preceding two sections in order to prove modular relations for the Ramanujan–Göllnitz–Gordon continued fraction, $K(q)$.

The following theorem is new.

Theorem 8.1. *Let $u=K(q)$ and $v=K(q^3)$. Then*

(i) $$v-u^3=3q^{5/2}\frac{f^3(-q^{24})}{f^3(-q^8)}\frac{\psi(-q^2)\phi(-q^4)}{\psi(-q)\psi(-q^3)}\frac{T(q)}{S^2(q)S(q^3)},$$

(ii) $$1+uv^3=\frac{f^3(-q^8)}{f^3(-q^{24})}\frac{\psi(-q^6)\phi(-q^{12})}{\psi(-q)\psi(-q^3)}\frac{T(q^3)}{S(q)S^2(q^3)},$$

(iii) $$v+u=q^{1/2}\frac{\phi(-q^4)\psi(-q^6)}{\psi(-q)\psi(-q^3)}\frac{1}{S(q)S(q^3)},$$

(iv) $$1-uv=\frac{\psi(-q^2)\phi(-q^{12})}{\psi(-q)\psi(-q^3)}\frac{1}{S(q)S(q^3)}.$$

Proof. Multiplying the identity in Theorem 6.1(i) by $q^{3/2}/(S^3(q)S(q^3))$ and applying (1.6), we deduce (i). By analogous arguments, (ii), (iii), and (iv) follow from (1.6) along with, respectively, Theorem 6.1(ii), Theorem 7.1(i), (ii). □

The next result, Theorem 8.2, is an analogue of Ramanujan's modular identity in Theorem 5.2. Chan and Huang [16] offered a systematic study of modular relations for the Ramanujan–Göllnitz–Gordon continued fraction, and gave the first proof of Theorem 8.2. Using theta function identities, Vasuki and Srivatsa Kumar [37] offered a new proof, and found further relations for $K(q)$ as well. Recently, Cho, Koo, and Park [18] found a new proof as part of their study of modular relations for $K(q)$ from the viewpoint of the theory of modular forms. Our proof utilizes Theorem 8.1. We remark that Theorem 8.1 is the first theorem to provide identities for the four factors in parentheses in Theorem 8.2.

Theorem 8.2. *Let u and v be as in Theorem 8.1. Then*

$$\left(v-u^3\right)\left(1+uv^3\right)=3uv(1-uv)(u+v).$$

Proof. Applying Theorem 8.1 and (1.6), we conclude that

$$\begin{aligned}\left(v-u^3\right)\left(1+uv^3\right)&=3q^{5/2}\frac{\psi(-q^2)\phi(-q^4)\psi(-q^6)\phi(-q^{12})}{\psi^2(-q)\psi^2(-q^3)}\frac{T(q)T(q^3)}{S^3(q)S^3(q^3)}\\&=3\left(\frac{q^2T(q)T(q^3)}{S(q)S(q^3)}\right)\left(\frac{\psi(-q^2)\phi(-q^{12})}{\psi(-q)\psi(-q^3)}\frac{1}{S(q)S(q^3)}\right)\\&\quad\times\left(q^{1/2}\frac{\phi(-q^4)\psi(-q^6)}{\psi(-q)\psi(-q^3)}\frac{1}{S(q)S(q^3)}\right)\\&=3uv(1-uv)(u+v).\end{aligned}$$

This completes the proof. □

The following identities are new.

Theorem 8.3. *Let u and v be as in Theorem* 8.1. *Then*

(i) $$\frac{v-u^3}{1+uv^3}=3q^{7/2}\frac{u}{v}\cdot\frac{f^6(-q^{24})}{f^6(-q^8)}\frac{\psi(-q^2)\phi(-q^4)}{\psi(-q^6)\phi(-q^{12})},$$

(ii) $$\frac{u^3}{v}+\frac{v}{u^3}=2+9q^2\frac{f^6(-q^{24})}{f^6(-q^8)}\cdot\frac{\psi^2(-q^2)\phi^2(-q^4)}{\psi^2(-q)\psi^2(-q^3)S(q)T(q)S(q^3)T(q^3)},$$

(iii) $$\frac{1}{uv^3}+uv^3=-2+\frac{f^6(-q^8)}{q^5f^6(-q^{24})}\cdot\frac{\psi^2(-q^6)\phi^2(-q^{12})}{\psi^2(-q)\psi^2(-q^3)S(q)T(q)S(q^3)T(q^3)}.$$

Proof. In Theorem 8.1, divide (i) by (ii). Simplifying with the help of (1.6), we complete the proof of (i). Identities (ii) and (iii) follow from parts (i) and (ii), respectively, of Theorem 8.1. The proofs are analogous to the proofs of Theorem 5.1(iv), (v); we omit the details. □

We remark that, with the use of Theorems 6.1, 7.1, (1.6), and Lemma 2.1, many further identities connecting $u=K(q)$ and $v=K(q^3)$ may be developed. We illustrate with one further theorem. The proofs are analogous to those of the preceding theorems, and so we omit the details.

Theorem 8.4. *With u and v as in Theorem 8.1, we have*

$$\text{(i)} \qquad \frac{u+v}{1-uv} = \sqrt{q}\frac{\phi(-q^4)\psi(-q^6)}{\phi(-q^{12})\psi(-q^2)},$$

$$\text{(ii)} \qquad \frac{u-v}{1+uv} = \sqrt{q}\frac{\phi(-q)\psi(q^6)}{\phi(-q^3)\psi(q^2)},$$

$$\text{(iii)} \qquad \frac{u^2-v^2}{1-u^2v^2} = q\frac{\phi(-q)\psi(q^{12})}{\phi(-q^3)\psi(q^4)},$$

$$\text{(iv)} \qquad \frac{u-v}{u+v} = \frac{\phi(-q)\phi(-q^{12})}{\phi(-q^4)\phi(-q^6)},$$

$$\text{(v)} \qquad \frac{1-uv}{1+uv} = \frac{\phi(-q^2)\phi(-q^{12})}{\phi(-q^3)\phi(-q^4)}.$$

9. Applications to the theory of partitions

The identities in Theorems 4.1, 6.1, and 7.1 all have applications in the theory of partitions. We illustrate this with certain representative examples, and remark that further theorems in this spirit may be similarly derived.

To describe these applications, we require the notion of colored partitions. We say that a positive integer n has k colors if there are k copies of n available and all of them are viewed as distinct objects. Partitions of positive integers into parts with colors are called *colored partitions*. For example, if 1 is allowed to have two colors, say red (r) and green (g), then the colored partitions of 2 are $2, 1_r+1_r$, $1_g+1_g, 1_r+1_g$. An important fact is that

$$\frac{1}{(q^r;q^s)_\infty^k}$$

is the generating function for the number of partitions of n, where all the parts are congruent to r $(\text{mod } s)$ and have k colors.

We next introduce notation and lemmas that are useful for extracting partition results from the modular relations that we consider. Let $p_e(n)$ denote the number of partitions of n into an even number of parts, and let $p_o(n) := p(n) - p_e(n)$ denote the number of partitions of n into an odd number of parts, where $p(n)$ is the ordinary partition function. Set $p(0) = 1$.

Lemma 9.1. *The following identities hold*:

$$\frac{1}{(-q;q)_\infty} = 1 + \sum_{n=1}^{\infty}\big(p_e(n) - p_o(n)\big)q^n, \tag{9.1}$$

$$p_e(n) = \sum_{0\leqslant j\leqslant\sqrt{n}} (-1)^j p\big(n-j^2\big), \quad n = 1, 2, \ldots. \tag{9.2}$$

Proof. For details, see [20, pp. 38–39, Eqs. (22.14), (22.21)]. □

Lemma 9.2. *(See [5, p. 37, Eq. (22.3)].) We have*

$$(-q;q)_\infty = \frac{1}{(q;q^2)_\infty}. \tag{9.3}$$

Equality (9.3) is the analytic equivalent of Euler's famous theorem: the number of partitions of a positive integer n into distinct parts is equal to the number of partitions of n into odd parts.

For simplicity, in this section we employ the standard notation

$$(a_1, a_2, \dots, a_n; q^t)_\infty := \prod_{j=1}^{n} (a_j; q^t)_\infty$$

and, for positive integers r and s with $r < s$,

$$(q^{r\pm}; q^s)_\infty := (q^r, q^{s-r}; q^s)_\infty.$$

Our first result is a consequence of Theorem 4.1(i).

Theorem 9.3. *Let $p_1(n)$ denote the number of partitions of n into parts congruent to $\pm 1 \pmod 5$, where the parts congruent to $\pm 1, \pm 4 \pmod{15}$ have three colors and the parts congruent to $\pm 6 \pmod{15}$ have four colors.*

Let $p_2(n)$ denote the number of partitions of n into parts congruent to $\pm 2 \pmod 5$, where the parts congruent to $\pm 2, \pm 7 \pmod{15}$ have three colors and the parts congruent to $\pm 3 \pmod{15}$ have four colors.

Let $p_3(n)$ denote the number of partitions of n into parts not divisible by 15, and with parts congruent to $\pm 3, \pm 5, \pm 6 \pmod{15}$ having two colors.

Define $p_i(0) := 1$ for $i = 1, 2, 3$, and $p_3(n) := 0$ for $n < 0$. Then, for each nonnegative integer n,

$$p_1(n) - p_2(n) = 3p_3(n-1).$$

Proof. Using (1.2) and (2.9), we write each of the functions appearing in Theorem 4.1(i) in terms of its product representation. Therefore,

$$\frac{1}{(q^{1\pm}; q^5)_\infty^3 (q^{6\pm}; q^{15})_\infty} - \frac{1}{(q^{2\pm}; q^5)_\infty^3 (q^{3\pm}; q^{15})_\infty} = \frac{3q(q^{15}; q^{15})_\infty^3}{(q; q)_\infty (q^3; q^3)_\infty (q^5; q^5)_\infty}. \tag{9.4}$$

Write each of the products on the right-hand side of (9.4) in the common base q^{15}. For example, $(q^3; q^3)_\infty = (q^3, q^6, q^9, q^{12}, q^{15}; q^{15})_\infty = (q^{3\pm}, q^{6\pm}, q^{15}; q^{15})_\infty$. Simplifying the result, we thus deduce that

$$\begin{aligned} &\frac{1}{(q^{1\pm}; q^5)_\infty^3 (q^{6\pm}; q^{15})_\infty} - \frac{1}{(q^{2\pm}; q^5)_\infty^3 (q^{3\pm}; q^{15})_\infty} \\ &\quad = \frac{3q}{(q^{1\pm}, q^{2\pm}, q^{3\pm}, q^{3\pm}, q^{4\pm}, q^{5\pm}, q^{5\pm}, q^{6\pm}, q^{6\pm}, q^{7\pm}; q^{15})_\infty}. \end{aligned} \tag{9.5}$$

Observe that the quotients on the left-hand side of (9.5) represent the generating functions for $p_1(n)$ and $p_2(n)$, and the right-hand side represents $3q$ times the generating function for $p_3(n)$. Hence, (9.5) is equivalent to

$$\sum_{n=0}^{\infty} p_1(n) q^n - \sum_{n=0}^{\infty} p_2(n) q^n = 3q \sum_{n=0}^{\infty} p_3(n) q^n. \tag{9.6}$$

The desired equality follows from equating coefficients on both sides of (9.6). $\square$

From Theorem 9.3, we immediately obtain the following corollary.

Corollary 9.4. *Let $p_1(n)$ and $p_2(n)$ be as defined in Theorem* 9.3. *For each nonnegative integer n,*

$$p_1(n) \equiv p_2(n) \pmod{3}. \tag{9.7}$$

It would be interesting to find a combinatorial explanation for (9.7).

Example. We illustrate Theorem 9.3 in the case $n=3$. Let the colors available be red (r), green (g), orange (o), and violet (v) when four are available, and red (r), green (g), and orange (o) when three are available. Then $p_1(3)=10$, $p_2(3)=4$, $p_3(2)=2$, and the required partitions are:

$$\begin{aligned}
p_1(3)\colon\quad & 1_r+1_r+1_r=1_g+1_g+1_g=1_o+1_o+1_o=1_r+1_r+1_g\\
&=1_r+1_r+1_o=1_g+1_g+1_r=1_g+1_g+1_o=1_o+1_o+1_r\\
&=1_o+1_o+1_g=1_r+1_g+1_o,\\
p_2(3)\colon\quad & 3_r=3_g=3_o=3_v,\\
p_3(2)\colon\quad & 2=1+1.
\end{aligned}$$

Theorem 9.5. *Let $p_1(n)$ denote the number of partitions of n where odd parts are distinct and congruent to* 3 (mod 6) *or* $\pm5, \pm11$ (mod 24), *and even parts are congruent to* 2 (mod 4) *or* 12 (mod 24).

Let $p_2(n)$ denote the number of partitions of n where odd parts are distinct and congruent to 3 (mod 6) *or* $\pm1, \pm7$ (mod 24), *and even parts satisfy the same conditions as for $p_1(n)$.*

Set $p_i(0) := 1$ for $i = 1, 2$, and $p_2(n) := 0$ for $n < 0$.

Then, for any positive integer n,

$$p_1(n) = \begin{cases} p_2(n-2), & \text{if } 12 \nmid n,\\ p_2(n-2) + 2\sum'_{0\leqslant j\leqslant\sqrt{k}}(-1)^j p(k-j^2), & \text{if } n = 12k,\end{cases}$$

where $\sum'$ indicates that the term corresponding to $j = 0$, i.e., $p(k)$, is weighted by $1/2$.

Proof. Apply (2.16) to the identity in Theorem 7.1(ii). Next, apply (2.6) on the left-hand side and (2.7) and (2.8) on the right-hand side of the resulting equation to deduce, after simplification, that

$$\left(q^3;q^6\right)_\infty\left(q^{5\pm},q^{11\pm};q^{24}\right)_\infty - q^2\left(q^3;q^6\right)_\infty\left(q^{1\pm},q^{7\pm};q^{24}\right)_\infty = \frac{(q^4;q^8)_\infty}{(-q^2;q^4)_\infty}\left(q^{12};q^{24}\right)_\infty^2. \tag{9.8}$$

Utilizing that

$$\frac{(q^4;q^8)_\infty}{(-q^2;q^4)_\infty} = \frac{(q^2;q^4)_\infty(-q^2;q^4)_\infty}{(-q^2;q^4)_\infty} = \left(q^2;q^4\right)_\infty,$$

we have, after replacing q by $-q$ in (9.8) and simplifying with the help of (9.3),

$$\frac{(-q^3;q^6)_\infty(-q^{5\pm},-q^{11\pm};q^{24})_\infty}{(q^2;q^4)_\infty(q^{12};q^{24})_\infty} - q^2\frac{(-q^3;q^6)_\infty(-q^{1\pm},-q^{7\pm};q^{24})_\infty}{(q^2;q^4)_\infty(q^{12};q^{24})_\infty} = \frac{1}{(-q^{12};q^{12})_\infty}.$$

Hence, by (9.1),

$$\sum_{n=0}^{\infty} p_1(n)q^n - q^2 \sum_{n=0}^{\infty} p_2(n)q^n = 1 + \sum_{n=1}^{\infty} \bigl(p_e(n) - p_o(n)\bigr) q^{12n}. \tag{9.9}$$

Equating coefficients on both sides of (9.9), we deduce that

$$p_1(n) - p_2(n-2) = 0, \quad \text{if } 12 \nmid n, \tag{9.10}$$

and

$$p_1(n) - p_2(n-2) = p_e(k) - p_o(k), \quad \text{if } n = 12k. \tag{9.11}$$

Moreover, by the definition of $p_o(n)$ and by (9.2),

$$\begin{aligned} p_e(k) - p_o(k) &= -p(k) + 2 \sum_{0 \leqslant j \leqslant \sqrt{k}} (-1)^j p\bigl(k - j^2\bigr) \\ &= 2 \sideset{}{'}\sum_{0 \leqslant j \leqslant \sqrt{k}} (-1)^j p\bigl(k - j^2\bigr). \end{aligned} \tag{9.12}$$

Combining (9.10)–(9.12), we complete the proof. □

Example. We verify Theorem 9.5 in the case $n = 13$. Then, $p_1(13) = 8$, $p_2(11) = 8$, and the required partitions are:

$$\begin{aligned} p_1(13){:}\quad & 13 = 11 + 2 = 10 + 3 = 9 + 2 + 2 = 6 + 5 + 2 = 6 + 3 + 2 + 2 \\ & \quad = 5 + 2 + 2 + 2 + 2 = 3 + 2 + 2 + 2 + 2 + 2, \\ p_2(11){:}\quad & 10 + 1 = 9 + 2 = 7 + 3 + 1 = 7 + 2 + 2 = 6 + 3 + 2 \\ & \quad = 6 + 2 + 2 + 1 = 3 + 2 + 2 + 2 + 2 = 2 + 2 + 2 + 2 + 2 + 1. \end{aligned}$$

Each of the modular identities that we have considered can yield several theorems in the theory of partitions. We illustrate by recording one further consequence of Theorem 7.1(ii).

Theorem 9.6. *Let $p_1(n)$ denote the number of partitions of n with odd parts congruent to $\pm 1, \pm 7 \pmod{24}$, and with even parts congruent to $12 \pmod{24}$ and having two colors.*

Let $p_2(n)$ denote the number of partitions of n with odd parts congruent to $\pm 5, \pm 11 \pmod{24}$, and with even parts satisfying the same conditions as for $p_1(n)$.

Let $\bar{p}(n)$ denote the number of partitions of n into distinct odd parts.

Define $p_i(0) := \bar{p}(0) := 1$ for $i = 1, 2$, and $p_2(n) := 0$ for $n < 0$. Then, for any nonnegative integer n,

$$p_1(n) - p_2(n-2) = \bar{p}(n).$$

Proof. We write Theorem 7.1(ii) in the equivalent form

$$\frac{1}{(q^{1\pm}, q^{7\pm}; q^{24})_\infty (q^{12}; q^{24})_\infty^2} - \frac{q^2}{(q^{5\pm}, q^{11\pm}; q^{24})_\infty (q^{12}; q^{24})_\infty^2} = \bigl(-q; q^2\bigr)_\infty,$$

whence the required equality follows. □

Example. We verify Theorem 9.6 in the case $n=13$. Let the colors available be red (r) and green (g). Then, $p_1(13)=4$, $p_2(11)=1$, $\bar{p}(13)=3$, and the required partitions are:

$$\begin{aligned}
p_1(13)\colon\quad & 12_r+1=12_g+1=7+1+1+1+1+1+1\\
&\quad =1+1+1+1+1+1+1+1+1+1+1+1+1,\\
p_2(11)\colon\quad & 11,\\
\bar{p}(13)\colon\quad & 13=9+3+1=7+5+1.
\end{aligned}$$

Acknowledgment

The author would like to thank Bruce Berndt for helpful comments and advice.

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